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ELECTROMAGNETIC EFFECTS IN THE THEORY OF RODS

By A. E. GREEN†, F.R.S., and P. M. NAGHDI‡
† *Mathematical Institute, Oxford University*, 24–29 *St Giles, Oxford OX*1 3*LB, U.K.*
‡ *Department of Mechanical Engineering, University of California, Berkeley, California* 94720, *U.S.A.*

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CONTENTS

This paper is concerned with the nonlinear and linear thermomechanical theories of deformable rod-like bodies in which account is taken of electromagnetic effects. The development is made by a direct approach with the use of the one-dimensional formulation of a theory of directed media called a *Cosserat curve*. The first part of the paper deals with the formulation of appropriate nonlinear equations governing the motion of a rod in the presence of electromagnetic and thermal effects. In the second part of the paper, emphasis is placed on the linearized version of the theory, a general discussion of the linear constitutive equations and determination of the constitutive coefficients, along with applications in a number of special cases including a magnetic thermoelastic rod and a non-conducting rod in free space.

1. Introduction

Electrodynamics of continua is a subject of considerable importance, with applications to both solids and fluids. As with continuum thermomechanics, when electromagnetic effects are absent, considerable difficulties present themselves in application of the three-dimensional theory to bodies with special geometrical features such as shells and rods. Usually some procedure is then introduced to reduce the theory to two-dimensional form for shells (and plates) and to one-dimensional form for rods (and beams). In the presence of electromagnetic effects, no general theory seems to be available for rods, although some work has been carried out for special problems in the linear piezoelectric theory of rods by Mindlin (1976). Previously, Green & Naghdi (1983, 1984) have developed a general theory of shells in the presence of electromagnetic effects and have applied the theory to a number of special cases involving both finite and small deformations. The purpose of the present paper is to discuss the corresponding developments for rods. Thus, using a *direct* approach based on a one-dimensional continuum model known as a *Cosserat* (or a *directed*) *curve*, we discuss here nonlinear and linear theories of deformable rod-like bodies in which full account is taken of both electromagnetic and thermal effects. The one-dimensional continuum model, designated as $\mathscr{R}$, comprises a material curve $\mathscr{L}$ embedded in a Euclidean 3-space together with two deformable vector fields – called directors – attached to every point of the material curve of $\mathscr{R}$. The directors, which are not necessarily along the unit principal normal, and the unit binormal vectors to the material curve have, in particular, the property that they remain unaltered in length under superposed rigid body motions. The body $\mathscr{R}$ with two directors is the simplest model for the construction of a general bending theory of slender rods. When the directors are absent, it represents a material curve appropriate for the construction of string theory. Theories which use more than two directors to represent various mechanical features of the rod-like body can be established similarly (see Naghdi 1979, §2), but the thermal and electromagnetic parts of the theory remain the same as those of the present paper.

The development of a complete theory of a Cosserat or a directed curve with two directors begins with a paper of Green & Laws (1966). A further development of the basic theory of a Cosserat curve along with certain general developments regarding the nonlinear and linear constitutive equations for elastic rods is contained in the work of Green *et al.* (1974). An account of the details of the basic theory is given in a recent paper by Naghdi (1982, part B), where additional relevant references on the subject can be found. The developments just referred to are made in the context of a thermomechanical theory of rods in which allowance is made for temperature changes only along some reference curve, such as the line of centroids, of the (three-dimensional) rod-like body. More recently, the scope of the thermomechanical theory of rods has been enlarged by Green & Naghdi (1979), who incorporated into the basic theory the effect of temperature changes in the cross-section of the rod. This development is achieved by means of an approach to thermomechanics in the three-dimensional theory introduced earlier (Green & Naghdi 1977), which provides a natural way of introducing any number of different one-dimensional temperature fields at each material point of material line $\mathscr{L}$ of $\mathscr{R}$.

We now turn to some background information concerning electromagnetic effects. At present, a (three-dimensional) theory of deformable media in the presence of electromagnetic effects may be developed at a number of levels of generality. One approach is based on a mixture theory in which the thermomechanical continuum is one constituent interacting with electric particle continua which, in turn, are acted upon by forces due to the electromagnetic fields. Another approach, adopted here, is to ignore details of the electric particle continua and consider only a single phase theory in which the continuum is acted upon directly by electromagnetic forces. We use an approximate non-relativistic theory in which the balance equations and also Maxwell's equations are invariant under a Galilean transformation of the form

$$\boldsymbol{r}^{*+} = \boldsymbol{Q}\boldsymbol{r}^{*} + \boldsymbol{\lambda}t, \quad t^{+} = t,$$

where $\boldsymbol{r}^*$ is the position vector of a material point of the body, t denotes time, the use of a superscript plus sign (as in $\boldsymbol{r}^{*+}$ and $\boldsymbol{t}^{+}$) refers to the corresponding quantities as a consequence of superposed rigid body motions, $\boldsymbol{\lambda}$ is a constant vector and $\boldsymbol{Q}$ is a constant orthogonal tensor. In addition, constitutive equations are to be unaltered by a constant superposed rigid body velocity and a constant superposed rigid body rotation. With these limitations, many authors have derived values for the three-dimensional electromagnetic force $\boldsymbol{f}_{\mathrm{e}}^{*}$, the electromagnetic couple $\boldsymbol{c}_{\mathrm{e}}^{*}$ and the rate of supply of electromagnetic energy w^*, and have discussed various constitutive relations. A survey of the various theories on the subject, together with extensive references, is given in a monograph by Hutter & van de Ven (1978). The survey of electromechanical interaction effects is presented in the form of five models, which the authors (Hutter & van de Ven 1978) refer to as the two-dipole models (i) and (ii), the Maxwell–Minkowski model, the statistical model and the Lorentz model. Although these models yield different values for the electromagnetic force, couple and rate of supply of electromagnetic energy, Hutter & van de Ven (1978) show that for a certain class of constitutive equations all theories are equivalent within the non-relativistic approximation. For our purpose, we select here three-dimensional values for $\boldsymbol{f}_{\mathrm{e}}^{*}$, $\boldsymbol{c}_{\mathrm{e}}^{*}$ and w^* which are a slight modification of the Maxwell–Minkowski model discussed by Hutter & van de Ven (1978). As will become evident, the theory for rods based on a Cosserat curve $\mathscr{R}$ (with two directors) will reflect the properties of this model, whose main equations are summarized in Appendixes A and B.

Specifically, the content of the paper is as follows. First, with reference to a Cosserat curve $\mathscr{R}$, in §2 the basic thermomechanical theory with extensions to electromagnetic effects is summarized and the consequences of the conservation laws in direct (coordinate free) notation are recorded in both spatial and material (or referential) forms. This is followed in §3 by appropriate electromagnetic balance equations for a moving rod-like body. These balance laws are analogues of corresponding conservation laws in the three-dimensional theory, which are summarized and discussed in Appendixes A and B. In §4, we consider constitutive equations for a magnetic, polarized thermoelastic Cosserat curve $\mathscr{R}$.

In the next four sections (§§5–8) emphasis is placed on the linear theory, although some of the developments, e.g. the first part of §6, are discussed in the context of the nonlinear theory. Thus, a linear theory of a thermoelastic magnetic Cosserat curve appropriate for a straight rod with a uniform cross-section is discussed in some detail in §5, with special cases elaborated upon further in §§6 and 7. The constitutive coefficients of the linearized theory of §5 are identified with the help of the results calculated in the context of the three-dimensional theory in Appendix C with the help of a number of results in Appendix B. The results of §6 are obtained in the spirit of a restricted theory (corresponding to the Bernoulli–Euler beam theory) and those of §7 are developed for a non-conducting rod in free space under isothermal conditions and in the absence of body force and applied tractions over the major surface of the rod. The linearized theory of §6 is further specialized and applied in §8 to isothermal forced vibrations of piezoelectric rods. Finally, the remaining two sections of the paper (§§9 and 10) briefly deal with an alternative representation of the theory for rods of rectangular cross-sections, which are particularly appropriate for elastic rectangular wave guides and piezoelectric rods.

2. Summary of thermomechanical theory with extensions to electromagnetic effects

We summarize the main kinematics and basic equations of the thermomechanical theory of a Cosserat curve based on the work of Green & Laws (1966) in the form developed by Green *et al.* (1974), Green & Naghdi (1979) and Naghdi (1982). Let the particles of the material curve $\mathscr{L}$ of the one-dimensional continuum $\mathscr{R}$ introduced in §1 be identified with a convected coordinate ζ and let the material curve in the present configuration at time t be referred to as c. Further, let $\boldsymbol{r}$ be the position vector of c, and $\boldsymbol{d}_\alpha$ $(\alpha = 1, 2)$ the directors at $\boldsymbol{r}$. A motion of $\mathscr{R}$ is then defined by

$$\boldsymbol{r} = \boldsymbol{r}(\zeta, t), \quad \boldsymbol{d}_\alpha = \boldsymbol{d}_\alpha(\zeta, t), \quad [\boldsymbol{d}_1\, \boldsymbol{d}_2\, \boldsymbol{a}_3] > 0, \tag{2.1}$$

where

$$\boldsymbol{a}_3 = \boldsymbol{a}_3(\zeta, t) = \partial \boldsymbol{r}/\partial\zeta \tag{2.2}$$

is a vector tangent to the curve c, and the directors remain unaltered in magnitude when the Cosserat curve is subject to superposed rigid body motions. The velocity and director velocities are given by

$$\boldsymbol{v} = \dot{\boldsymbol{r}}(\zeta, t), \quad \boldsymbol{w}_\alpha = \dot{\boldsymbol{d}}_\alpha(\zeta, t), \tag{2.3}$$

where a dot denotes a material time derivative with respect to t with ζ held fixed. In the reference configuration of $\mathscr{R}$, which we take to be the initial configuration, let the material curve $\mathscr{L}$ be referred to by C and denote the position vector of C by $\boldsymbol{R}$, the tangent vector to C by $\boldsymbol{A}_3$ and the initial directors by $\boldsymbol{D}_\alpha$. Then

$$\left.\begin{aligned} \boldsymbol{R} = \boldsymbol{R}(\zeta) = \boldsymbol{r}(\zeta, 0), \quad \boldsymbol{A}_3 = \boldsymbol{A}_3(\zeta) = \partial \boldsymbol{R}/\partial\zeta = \boldsymbol{a}_3(\zeta, 0), \\ \boldsymbol{D}_\alpha = \boldsymbol{D}_\alpha(\zeta) = \boldsymbol{d}_\alpha(\zeta, 0). \end{aligned}\right\} \tag{2.4}$$

We define a set of linearly independent vectors $\boldsymbol{d}_i$ and their reciprocals $\boldsymbol{d}^i (i = 1, 2, 3)$, and the corresponding values $\boldsymbol{D}_i$, $\boldsymbol{D}^i$ in the reference configuration, by the formulae

$$\boldsymbol{d}_3 = \boldsymbol{a}_3, \quad \boldsymbol{d}^i \cdot \boldsymbol{d}_j = \delta^i_j, \quad \boldsymbol{D}_3 = \boldsymbol{A}_3, \quad \boldsymbol{D}^i \cdot \boldsymbol{D}_j = \delta^i_j. \tag{2.5}$$

In addition, let $\boldsymbol{a}_i$ be a set of orthogonal base vectors and $\boldsymbol{a}^i$ their reciprocals such that $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{\nu}$ are unit vectors and

$$\left.\begin{gathered}\boldsymbol{a}^1 = \boldsymbol{a}_1, \quad \boldsymbol{a}^2 = \boldsymbol{a}_2, \quad a_{33} = \boldsymbol{a}_3\cdot\boldsymbol{a}_3, \quad \boldsymbol{a}_\alpha\cdot\boldsymbol{a}_\beta = \delta_{\alpha\beta}, \quad \boldsymbol{a}_\alpha\cdot\boldsymbol{a}_3 = 0,\\ \boldsymbol{\nu} = \boldsymbol{a}_3/a_{33}^{\frac{1}{2}} = \boldsymbol{a}^3 a_{33}^{\frac{1}{2}}, \quad \partial\boldsymbol{a}_i/\partial\zeta = \boldsymbol{C}\boldsymbol{a}_i = c_{ri}\boldsymbol{a}^r = c^r{}_i\,\boldsymbol{a}_r,\\ c_{11} = c_{22} = 0, \quad c_{12}+c_{21} = 0, \quad c_{\alpha 3}+c_{3\alpha} = 0, \quad \partial a_{33}/\partial\zeta = 2c_{33},\\ \boldsymbol{L} = \dot{\boldsymbol{a}}_1\otimes\boldsymbol{a}^i, \quad \dot{\boldsymbol{a}}_i = \boldsymbol{L}\boldsymbol{a}_i.\end{gathered}\right\} \tag{2.6}$$

The vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ may move with the curve in any way prescribed subject to (2.6), and they are functions of ζ, t. The corresponding values of $\boldsymbol{\nu}$, $\boldsymbol{a}_i$, $\boldsymbol{a}^i$ and $\boldsymbol{C}$ in the reference configuration are denoted by ${}_{\mathrm{R}}\boldsymbol{\nu}$, $\boldsymbol{A}_i$, $\boldsymbol{A}^i$ and ${}_{\mathrm{R}}\boldsymbol{C}$, respectively.

We now recall the equations of mass conservation, momentum, director momenta and moment of momentum, which in spatial form are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\zeta_1}^{\zeta_2}\rho\,\mathrm{d}s = 0, \quad \mathrm{d}s = a_{33}^{\frac{1}{2}}\,\mathrm{d}\zeta, \tag{2.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\zeta_1}^{\zeta_2}(\boldsymbol{v}+y^{0\beta}\boldsymbol{w}_\beta)\,\mathrm{d}s = \int_{\zeta_1}^{\zeta_2}\rho(\boldsymbol{f}+\boldsymbol{f}_{\mathrm{e}})\,\mathrm{d}s + [\boldsymbol{n}]_{\zeta_1}^{\zeta_2}, \tag{2.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\zeta_1}^{\zeta_2}\rho(y^{0\alpha}\boldsymbol{v}+y^{\alpha\beta}\boldsymbol{w}_\beta)\,\mathrm{d}s = \int_{\zeta_1}^{\zeta_2}\{\rho(\boldsymbol{l}^\alpha+\boldsymbol{l}^\alpha_{\mathrm{e}})-\boldsymbol{k}^{\alpha 3}\}\,\mathrm{d}s + [\boldsymbol{m}^\alpha]_{\zeta_1}^{\zeta_2}, \tag{2.9}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\zeta_1}^{\zeta_2}\rho\{\boldsymbol{r}\times(\boldsymbol{v}+y^{0\beta}\boldsymbol{w}_\beta)+\boldsymbol{d}_\alpha\times(y^{0\alpha}\boldsymbol{v}+y^{\alpha\beta}\boldsymbol{w}_\beta)\}\,\mathrm{d}s\\ = \int_{\zeta_1}^{\zeta_2}\rho\{\boldsymbol{r}\times(\boldsymbol{f}+\boldsymbol{f}_{\mathrm{e}})+\boldsymbol{d}_\alpha\times(\boldsymbol{l}^\alpha+\boldsymbol{l}^\alpha_{\mathrm{e}})+\boldsymbol{c}_{\mathrm{e}}\}\,\mathrm{d}s + [\boldsymbol{r}\times\boldsymbol{n}+\boldsymbol{d}_\alpha\times\boldsymbol{m}^\alpha]_{\zeta_1}^{\zeta_2}.\end{aligned} \tag{2.10}$$

In (2.7)–(2.10), $\rho = \rho(\zeta, t)$ is mass per unit length of c, $y^{\alpha\beta}$, $y^{0\alpha}$ are inertia coefficients which are functions of ζ, independent of t, $\boldsymbol{n} = \boldsymbol{n}(\zeta, t)$ is the contact force vector, $\boldsymbol{m}^\alpha = \boldsymbol{m}^\alpha(\zeta, t)$ are contact director force vectors, each a three-dimensional field in the present configuration, $\boldsymbol{f} = \boldsymbol{f}(\zeta, t)$ is the assigned force and $\boldsymbol{l}^\alpha = \boldsymbol{l}^\alpha(\zeta, t)$ are the assigned director forces, each a three-dimensional vector field per unit mass of c. Also, the quantities $\boldsymbol{f}_{\mathrm{e}}$, $\boldsymbol{l}^\alpha_{\mathrm{e}}$, $\boldsymbol{c}_{\mathrm{e}}$ are, respectively, the force vector, director force vector and (axial) couple vector per unit mass due to the electromagnetic fields, $\boldsymbol{k}^{\alpha 3}$ are the intrinsic forces (previously denoted by $a_{33}^{-\frac{1}{2}}\boldsymbol{k}^\alpha$), and on the right-hand side of (2.7)–(2.10) we have used the notation

$$[f(\zeta, t)]_{\zeta_1}^{\zeta_2} = f(\zeta_2, t) - f(\zeta_1, t). \tag{2.11}$$

The assigned field $\boldsymbol{f}$ represents the combined effect of (*a*) the stress vector on the major surface of the rod-like body denoted by $\boldsymbol{f}_{\mathrm{s}}$ and (*b*) an integrated contribution arising from three-dimensional body forces, denoted by $\boldsymbol{f}_{\mathrm{b}}$. A parallel statement holds for the assigned fields $\boldsymbol{l}^\alpha$ (see equations (B 5) of Appendix B). Therefore, we may write

$$\boldsymbol{f} = \boldsymbol{f}_{\mathrm{b}} + \boldsymbol{f}_{\mathrm{s}}, \quad \boldsymbol{l}^\alpha = \boldsymbol{l}^\alpha_{\mathrm{b}} + \boldsymbol{l}^\alpha_{\mathrm{s}}. \tag{2.12}$$

The force $\boldsymbol{f}_{\mathrm{e}}$, director forces $\boldsymbol{l}^\alpha_{\mathrm{e}}$ and couple $\boldsymbol{c}_{\mathrm{e}}$ are due to the integrated contributions from the three-dimensional electromagnetic fields, and it is convenient to keep these separate from (2.12); in this regard, see also equations (B 5) of Appendix B.

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Balance equations may be stated in material form by replacing ρ, $\mathrm{d}s$ and $\boldsymbol{k}^{\alpha 3}$ in (2.7)–(2.10) by ${}_{\mathrm{R}}\rho$, $\mathrm{d}_{\mathrm{R}}s = A_{33}^{\frac{1}{2}}\,\mathrm{d}\zeta$ and ${}_{\mathrm{R}}\boldsymbol{k}^{\alpha 3}$, respectively. With this in mind, it follows that field equations corresponding to (2.7)–(2.10), or to the material forms of these equations, are:

$$\left.\begin{aligned} \dot{\rho}+\rho\,\mathrm{div}_s\,\boldsymbol{v} = \dot{\rho}+\rho\boldsymbol{a}^3\cdot\partial\boldsymbol{v}/\partial\zeta = 0 \quad \text{or} \quad \rho a_{33}^{\frac{1}{2}} = {}_{\mathrm{R}}\rho A_{33}^{\frac{1}{2}} = \lambda,\\ \lambda(\dot{\boldsymbol{v}}+y^{0\beta}\,\dot{\boldsymbol{w}}_\beta) = \lambda(\boldsymbol{f}+\boldsymbol{f}_{\mathrm{e}})+\partial\boldsymbol{n}/\partial\zeta,\\ \lambda(y^{0\alpha}\,\dot{\boldsymbol{v}}+y^{\alpha\beta}\,\dot{\boldsymbol{w}}_\beta) = \lambda(\boldsymbol{l}^\alpha+\boldsymbol{l}^\alpha_{\mathrm{e}})-\boldsymbol{k}^\alpha+\partial\boldsymbol{m}^\alpha/\partial\zeta,\\ \lambda\boldsymbol{c}_{\mathrm{e}}+\boldsymbol{a}_3\times\boldsymbol{n}+\boldsymbol{d}_\alpha\times\boldsymbol{k}^\alpha+\frac{\partial\boldsymbol{d}_\alpha}{\partial\zeta}\times\boldsymbol{m}^\alpha = \boldsymbol{0}, \end{aligned}\right\} \tag{2.13}$$

where
$$\boldsymbol{k}^\alpha = \boldsymbol{k}^{\alpha 3}a_{33}^{\frac{1}{2}} = {}_{\mathrm{R}}\boldsymbol{k}^{\alpha 3}A_{33}^{\frac{1}{2}}. \tag{2.14}$$

The balances of entropy and energy for every part of the material curve c in the present configuration, with some change of notation from Green & Naghdi (1979) in order to simplify the representation of the various entropy and heat functions, are

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\zeta_1}^{\zeta_2}\rho\eta\,\mathrm{d}s = \int_{\zeta_1}^{\zeta_2}\rho(s+\xi)\,\mathrm{d}s-[k]_{\zeta_1}^{\zeta_2},\\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\zeta_1}^{\zeta_2}\rho\eta_{MN}\,\mathrm{d}s = \int_{\zeta_1}^{\zeta_2}\rho(s_{MN}+\xi_{MN})\,\mathrm{d}s-[k_{MN}]_{\zeta_1}^{\zeta_2}, \end{aligned}\right\} \tag{2.15}$$

for $M+N = K$, $K = 1, 2, \ldots, P$, and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\zeta_1}^{\zeta_2}&\{\epsilon+\tfrac{1}{2}(\boldsymbol{v}\cdot\boldsymbol{v}+2y^{0\alpha}\,\boldsymbol{v}\cdot\boldsymbol{w}_\alpha+y^{\alpha\beta}\,\boldsymbol{w}_\alpha\cdot\boldsymbol{w}_\beta)\}\,\rho\,\mathrm{d}s\\ &= \int_{\zeta_1}^{\zeta_2}\left\{r+\sum_{K=1}^{P}r_{MN}+(\boldsymbol{f}+\boldsymbol{f}_{\mathrm{e}})\cdot\boldsymbol{v}+(\boldsymbol{l}^\alpha+\boldsymbol{l}^\alpha_{\mathrm{e}})\cdot\boldsymbol{w}_\alpha+\bar{w}\right\}\rho\,\mathrm{d}s\\ &\qquad+\left[\boldsymbol{n}\cdot\boldsymbol{v}+\boldsymbol{m}^\alpha\cdot\boldsymbol{w}_\alpha-h-\sum_{K=1}^{P}h_{MN}\right]_{\zeta_1}^{\zeta_2}. \end{aligned} \tag{2.16}$$

The summation $\sum_{K=1}^{P}$ is over all values of $K = M+N$ where M, N are integers or zeros. In (2.15) and (2.16), η, η_{MN} are entropy densities, ϵ is internal energy density, k, k_{MN} are entropy fluxes, h, h_{MN} are heat fluxes, ξ, ξ_{MN} are internal rates of production of entropy, s, s_{MN} are external rates of supply of entropy, r, r_{MN} are external rates of supply of heat, and $r = \theta s$, $r_{MN} = \theta_{MN}\,s_{MN}$, $h = \theta k$, $h_{MN} = \theta_{MN}\,k_{MN}$, where $\theta > 0$ and θ_{MN} are temperatures, and there is no summation over M, N where they appear twice. The quantity $\bar{w}$ represents the rate of work of the electromagnetic couple $\boldsymbol{c}_{\mathrm{e}}$ together with the rate of supply of electromagnetic energy due to the electromagnetic fields. The external rates of supply of entropy s, s_{MN} consist of two parts, one due to the entropy supply across the major surface of the rod and the other due to integrated volume supplies of entropy through the rod, as indicated in (B 6).

Similar balance equations hold in material form by replacing ρ, $\mathrm{d}s$ in (2.15) and (2.16) by ${}_{\mathrm{R}}\rho$, $\mathrm{d}_{\mathrm{R}}s$. Field equations which correspond to (2.15), (2.16) or their material forms are

$$\left.\begin{aligned} \lambda\dot{\eta} = \lambda(s+\xi)-\partial k/\partial\zeta,\\ \lambda\dot{\eta}_{MN} = \lambda(s_{MN}+\xi_{MN})-\partial k_{MN}/\partial\zeta, \end{aligned}\right\} \tag{2.17}$$

and

$$\lambda(-\dot{\epsilon}+\theta\dot{\eta}+\sum\theta_{MN}\dot{\eta}_{MN})+\lambda\bar{w}-\lambda(\theta\xi+\sum\theta_{MN}\xi_{MN})+P-k\,\partial\theta/\partial\zeta-\sum k_{MN}\,\partial\theta_{MN}/\partial\zeta=0, \tag{2.18}$$

where
$$P=\boldsymbol{n}\cdot\partial\boldsymbol{v}/\partial\zeta+\boldsymbol{k}^{\alpha}\cdot\boldsymbol{w}_{\alpha}+\boldsymbol{m}^{\alpha}\cdot\partial\boldsymbol{w}_{\alpha}/\partial\zeta. \tag{2.19}$$

3. Electromagnetic equations for a Cosserat curve

To complete the system of equations for a rod by the direct approach, we must now introduce appropriate electromagnetic variables and equations. In doing this we are guided by the exact three-dimensional developments described in Appendix B. We assume that the electromagnetic effects are represented by the following fields:

$$\left.\begin{array}{l}\text{the electric field vectors: } \boldsymbol{e}^{*}_{MN}=e^{*}_{MNi}\,\boldsymbol{a}^{i},\\ \text{the electric displacement vectors: } \bar{\boldsymbol{d}}_{MN}=\bar{d}_{MN}{}^{i}\,\boldsymbol{a}_{i},\\ \text{the magnetic field vectors: } \boldsymbol{h}^{*}_{MN}=\boldsymbol{h}^{*}_{MNi}\,\boldsymbol{a}^{i},\\ \text{the magnetic induction vectors: } \boldsymbol{b}_{MN}=b_{MN}{}^{i}\,\boldsymbol{a}_{i},\\ \text{the current density vectors: } \boldsymbol{j}^{*}_{MN}=j^{*}_{MN}{}^{i}\,\boldsymbol{a}_{i},\\ \text{the free charges represented by the scalar fields: } e_{MN},\end{array}\right\} \tag{3.1}$$

for $M+N=K$, $K=0,1,\dots,L$ with M, N integers or zeros. We use an overbar to designate the electric displacement fields (rather than the more customary symbols without overbar) in order to avoid confusion with the notation for director fields such as $\boldsymbol{d}_{\alpha}$ in (2.2).

We need appropriate balance equations for a moving rod, which may be thought of as analogues of the three-dimensional balance equations associated with Faraday, Ampere and Gauss. For this we depend on the developments in Appendix B, particularly (B 13)–(B 16). Then, corresponding to the Gauss integral balance in the three dimensions we have Gauss-type balances for the rod in the forms

$$\left.\begin{aligned}\int_{\zeta_1}^{\zeta_2} b'_{MN}\,\mathrm{d}s+[\boldsymbol{b}_{MN}\cdot\boldsymbol{v}]_{\zeta_1}^{\zeta_2}&=\int_{\zeta_1}^{\zeta_2}\left(\sum_{K=0}^{M}\chi^{M}_{K}\,b_{KN}{}^{1}+\sum_{K=0}^{N}\bar{\chi}^{N}_{K}\,b_{MK}{}^{2}\right)\mathrm{d}s,\\ \int_{\zeta_1}^{\zeta_2} \mathrm{d}'_{MN}\,\mathrm{d}s+[\bar{\boldsymbol{d}}_{MN}\cdot\boldsymbol{v}]_{\zeta_1}^{\zeta_2}&=\int_{\zeta_1}^{\zeta_2}\left(e_{MN}+\sum_{K=0}^{M}\psi^{M}_{K}\,\bar{d}_{KN}{}^{1}+\sum_{K=0}^{N}\bar{\psi}^{N}_{K}\,\bar{d}_{MK}{}^{2}\right)\mathrm{d}s.\end{aligned}\right\} \tag{3.2}$$

Similarly, corresponding to the Faraday and Ampere integral balances we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\zeta_1}^{\zeta_2}\boldsymbol{b}_{MN}\,\mathrm{d}s=[\boldsymbol{e}^{*}_{MN}\times\boldsymbol{v}]_{\zeta_1}^{\zeta_2}+\int_{\zeta_1}^{\zeta_2}\left(\boldsymbol{L}\boldsymbol{b}_{MN}+\boldsymbol{a}^{1}\times\sum_{K=0}^{M}\chi^{M}_{K}\,\boldsymbol{e}^{*}_{KN}+\boldsymbol{a}^{2}\times\sum_{K=0}^{N}\bar{\chi}^{N}_{K}\,\boldsymbol{e}^{*}_{MK}\right)\mathrm{d}s\\ +\int_{\zeta_1}^{\zeta_2}\{\boldsymbol{e}'_{MN}-\boldsymbol{C}\boldsymbol{a}_{\alpha}[\boldsymbol{a}^{\alpha}\boldsymbol{e}^{*}_{MN}\,\boldsymbol{a}^{3}]\}\,\mathrm{d}s,\end{aligned} \tag{3.3}$$

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\zeta_1}^{\zeta_2}\bar{\boldsymbol{d}}_{MN}\,\mathrm{d}s=[\boldsymbol{h}^{*}_{MN}\times\boldsymbol{v}]_{\zeta_1}^{\zeta_2}+\int_{\zeta_1}^{\zeta_2}\left(\boldsymbol{j}^{*}_{MN}-\boldsymbol{L}\bar{\boldsymbol{d}}_{MN}+\boldsymbol{a}^{1}\times\sum_{K=0}^{M}\psi^{M}_{K}\,\boldsymbol{h}^{*}_{KN}\right.\\ \left.+\boldsymbol{a}^{2}\times\sum_{K=0}^{N}\bar{\psi}^{N}_{K}\,\boldsymbol{h}^{*}_{MK}\right)\mathrm{d}s+\int_{\zeta_1}^{\zeta_2}(\boldsymbol{h}'_{MN}-\boldsymbol{C}\boldsymbol{a}_{\alpha}[\boldsymbol{a}^{\alpha}\,\boldsymbol{h}^{*}_{MN}\,\boldsymbol{a}^{3}])\,\mathrm{d}s\end{aligned} \tag{3.4}$$

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for $M+N=K$, $K=0,1,\ldots,L$, where b'_{MN}, d'_{MN}, $\boldsymbol{e}'_{MN}$, $\boldsymbol{h}'_{MN}$, due to contributions from the major surface of the rod, are given by (B 22)–(B 24) in Appendix B. Similar balance equations may be written in material form in terms of vectors and scalars:

$$\left.\begin{aligned}\boldsymbol{E}_{MN} = E_{MNi}\boldsymbol{A}^i,\quad \boldsymbol{H}_{MN} = H_{MNi}\boldsymbol{A}^i,\quad \bar{\boldsymbol{D}}_{MN} = \bar{D}_{MN}{}^i\boldsymbol{A}_i,\quad \boldsymbol{B}_{MN} = B_{MN}{}^i\boldsymbol{A}_i,\\ \boldsymbol{J}_{MN} = J_{MN}{}^i\boldsymbol{A}_i,\quad E_{MN},\end{aligned}\right\} \tag{3.5}$$

where these functions are related to those in (3.1) by equations (B 25) in Appendix B. Field equations derived from (3.2)–(3.4) are

$$-b'_{MN}+\sum_{K=0}^{M}\chi_K^M b_{KN}{}^1+\sum_{K=0}^{N}\bar{\chi}_K^N b_{MK}{}^2 = \mathrm{div}_c(b_{MN}{}^3\boldsymbol{a}_3) = a_{33}^{-\frac{1}{2}}\partial(b_{MN}{}^3a_{33}^{\frac{1}{2}})/\partial\zeta, \tag{3.6}$$

$$-d'_{MN}+e_{MN}+\sum_{K=0}^{M}\psi_K^M \bar{d}_{KN}{}^1+\sum_{K=0}^{N}\bar{\psi}_K^N \bar{d}_{MK}{}^2 = \mathrm{div}_c(\bar{d}_{MN}{}^3\boldsymbol{a}_3) = a_{33}^{-\frac{1}{2}}\partial(\bar{d}_{MN}{}^3a_{33}^{\frac{1}{2}})/\partial\zeta, \tag{3.7}$$

$$\begin{aligned}\dot{\boldsymbol{b}}_{MN}+\boldsymbol{b}_{MN}\,\mathrm{div}_c\boldsymbol{v}-\boldsymbol{L}\boldsymbol{b}_{MN} &= -\mathrm{curl}_c\boldsymbol{e}^*_{MN}+\boldsymbol{a}^3\times\boldsymbol{C}\boldsymbol{e}^*_{MN}+\boldsymbol{e}'_{MN}\\ &\quad+\boldsymbol{a}^1\times\sum_{K=0}^{M}\chi_K^M\boldsymbol{e}^*_{KN}+\boldsymbol{a}^2\times\sum_{K=0}^{N}\bar{\chi}_K^N\boldsymbol{e}^*_{MK},\end{aligned} \tag{3.8}$$

$$\begin{aligned}-(\dot{\bar{\boldsymbol{d}}}_{MN}+\bar{\boldsymbol{d}}_{MN}\,\mathrm{div}_c\boldsymbol{v}-\boldsymbol{L}\bar{\boldsymbol{d}}_{MN}) &= -\mathrm{curl}_c\boldsymbol{h}^*_{MN}+\boldsymbol{a}^3\times\boldsymbol{C}\boldsymbol{h}^*_{MN}+\boldsymbol{j}^*_{MN}+\boldsymbol{h}'_{MN}\\ &\quad+\boldsymbol{a}^1\times\sum_{K=0}^{M}\psi_K^M\boldsymbol{h}^*_{KN}+\boldsymbol{a}^2\times\sum_{K=0}^{N}\bar{\psi}_K^N\boldsymbol{h}^*_{MK},\end{aligned} \tag{3.9}$$

where
$$\mathrm{curl}_c(\ ) = \boldsymbol{a}^3\times\partial(\ )/\partial\zeta. \tag{3.10}$$

Similarly, from balance equations in material form we have

$$-B'_{MN}+\sum_{K=0}^{M}\chi_K^M B_{KN}{}^1+\sum_{K=0}^{N}\bar{\chi}_K^N B_{MK}{}^2 = \mathrm{Div}_c(B_{MN}{}^3\boldsymbol{A}_3) = A_{33}^{-\frac{1}{2}}\partial(B_{MN}{}^3A_{33}^{\frac{1}{2}})/\partial\zeta, \tag{3.11}$$

$$-D'_{MN}+E_{MN}+\sum_{K=0}^{M}\psi_K^M \bar{D}_{KN}{}^1+\sum_{K=0}^{N}\bar{\psi}_K^N \bar{D}_{MK}{}^2 = \mathrm{Div}_c(\bar{D}_{MN}{}^3\boldsymbol{A}_3) = A_{33}^{-\frac{1}{2}}\partial(\bar{D}_{MN}{}^3A_{33}^{\frac{1}{2}})/\partial\zeta, \tag{3.12}$$

$$\dot{\boldsymbol{B}}_{MN} = -\mathrm{Curl}_c\boldsymbol{E}_{MN}+\boldsymbol{A}^3\times{}_R\boldsymbol{C}\boldsymbol{E}_{MN}+\boldsymbol{E}'_{MN}+\boldsymbol{A}^1\times\sum_{K=0}^{M}\chi_K^M\boldsymbol{E}_{KN}+\boldsymbol{A}^2\times\sum_{K=0}^{N}\bar{\chi}_K^N\boldsymbol{E}_{MK}, \tag{3.13}$$

$$\begin{aligned}-\dot{\bar{\boldsymbol{D}}}_{MN} &= -\mathrm{Curl}_c\boldsymbol{H}_{MN}+\boldsymbol{A}^3\times{}_R\boldsymbol{C}\boldsymbol{H}_{MN}+\boldsymbol{J}_{MN}+\boldsymbol{H}'_{MN}\\ &\quad+\boldsymbol{A}^1\times\sum_{K=0}^{M}\psi_K^M\boldsymbol{H}_{KN}+\boldsymbol{A}^2\times\sum_{K=0}^{N}\bar{\psi}_K^N\boldsymbol{H}_{MK},\end{aligned} \tag{3.14}$$

where
$$\mathrm{Curl}_c(\ ) = \boldsymbol{A}^3\times\partial(\ )/\partial\zeta. \tag{3.15}$$

To complete the theory in §§2 and 3, we need explicit expressions for $\bar{w}$, $\boldsymbol{f}_{\mathrm{e}}$, $\boldsymbol{l}_{\mathrm{e}}^{\alpha}$, $\boldsymbol{c}_{\mathrm{e}}$. In view of (B 5), (B 29) and (B 30) of Appendix B, we adopt the following values for $\bar{w}$ and $\boldsymbol{c}_{\mathrm{e}}$:

$$\left.\begin{aligned}\lambda\bar{w} &= P_{\mathrm{e}}+\sum_{M+N=0}^{L}(\hat{\boldsymbol{J}}_{MN}\cdot\tilde{\boldsymbol{E}}_{MN}+\tilde{\boldsymbol{E}}_{MN}\cdot\dot{\hat{\boldsymbol{D}}}_{MN}+\tilde{\boldsymbol{H}}_{MN}\cdot\dot{\hat{\boldsymbol{B}}}_{MN}),\\ \lambda\boldsymbol{c}_{\mathrm{e}} &= \boldsymbol{a}_3\times\boldsymbol{n}_{\mathrm{e}}+\boldsymbol{d}_{\alpha}\times\boldsymbol{k}_{\mathrm{e}}^{\alpha}+\frac{\partial\boldsymbol{d}_{\alpha}}{\partial\zeta}\times\boldsymbol{m}_{\mathrm{e}}^{\alpha},\\ P_{\mathrm{e}} &= \boldsymbol{n}_{\mathrm{e}}\cdot\partial\boldsymbol{v}/\partial\zeta+\boldsymbol{k}_{\mathrm{e}}^{\alpha}\cdot\boldsymbol{w}_{\alpha}+\boldsymbol{m}_{\mathrm{e}}^{\alpha}\cdot\partial\boldsymbol{w}_{\alpha}/\partial\zeta,\end{aligned}\right\} \tag{3.16}$$

where
$$\left.\begin{aligned}\tilde{\boldsymbol{E}}_{MN} = \tilde{E}_{MNi}\boldsymbol{A}^i, \quad \tilde{\boldsymbol{H}}_{MN} = \tilde{H}_{MNi}\boldsymbol{A}^i, \quad \hat{\boldsymbol{D}}_{MN} = \hat{D}_{MN}{}^i\boldsymbol{A}_i,\\ \hat{\boldsymbol{B}}_{MN} = \hat{B}_{MN}{}^i\boldsymbol{A}_i, \quad \hat{\boldsymbol{J}}_{MN} = \hat{J}_{MN}{}^i\boldsymbol{A}_i\end{aligned}\right\} \tag{3.17}$$

and
$$E_{MNi} = \sum_{R+S=0}^{L} I_{(i)}^{MNRS}\,\tilde{E}_{RSi}, \quad H_{MNi} = \sum_{R+S=0}^{L} K_{(i)}^{MNRS}\,\tilde{H}_{RSi}. \tag{3.18}$$

The coefficients in (3.18) are given by (B 28).

Under a Galilean transformation of the vector $\boldsymbol{r}$, directors $\boldsymbol{d}_\alpha$ and vectors $\boldsymbol{a}_i$, namely

$$\boldsymbol{r}^+ = \boldsymbol{Q}\boldsymbol{r}+\boldsymbol{r}_0\,t, \quad \boldsymbol{d}_\alpha^+ = \boldsymbol{Q}\boldsymbol{d}_\alpha, \quad \boldsymbol{a}_i^+ = \boldsymbol{Q}\boldsymbol{a}_i, \tag{3.19}$$

where $\boldsymbol{r}_0$ is a constant vector and $\boldsymbol{Q}$ is a constant orthogonal tensor, the following transformations hold for the electromagnetic quantities:

$$\left.\begin{array}{lll}\bar{\boldsymbol{d}}_{MN}{}^+ = \boldsymbol{Q}\bar{\boldsymbol{d}}_{MN}, & \boldsymbol{b}_{MN}{}^+ = \boldsymbol{Q}\boldsymbol{b}_{MN}\det\boldsymbol{Q}, & \boldsymbol{e}^*_{MN}{}^+ = \boldsymbol{Q}\boldsymbol{e}^*_{MN},\\ \boldsymbol{h}^*_{MN}{}^+ = \boldsymbol{Q}\boldsymbol{h}^*_{MN}\det\boldsymbol{Q}, & \boldsymbol{j}^*_{MN}{}^+ = \boldsymbol{Q}\boldsymbol{j}^*_{MN}, & e_{MN}{}^+ = e_{MN},\\ \bar{\boldsymbol{D}}_{MN}{}^+ = \bar{\boldsymbol{D}}_{MN}\det\boldsymbol{Q}, & \boldsymbol{B}_{MN}{}^+ = \boldsymbol{B}_{MN}, & \boldsymbol{E}_{MN}{}^+ = \boldsymbol{E}_{MN},\\ \boldsymbol{H}_{MN}{}^+ = \boldsymbol{H}_{MN}\det\boldsymbol{Q}, & \boldsymbol{J}_{MN}{}^+ = \boldsymbol{J}_{MN}\det\boldsymbol{Q}, & E_{MN}{}^+ = E_{MN}\det\boldsymbol{Q}.\end{array}\right\} \tag{3.20}$$

When the Cosserat rod is subjected to a constant rigid body velocity and a constant rigid body rotation, the same relations (3.20) hold with $\det\boldsymbol{Q} = 1$.

4. Magnetic polarized thermoelastic Cosserat curve

We introduce the Helmholtz free energy function ψ for the Cosserat rod by the expression

$$\psi = \epsilon-\theta\eta-\sum\theta_{MN}\,\eta_{MN}-\lambda^{-1}\sum(\tilde{\boldsymbol{E}}_{MN}\cdot\hat{\boldsymbol{D}}_{MN}+\tilde{\boldsymbol{H}}_{MN}\cdot\hat{\boldsymbol{B}}_{MN}), \tag{4.1}$$

where the vectors $\tilde{\boldsymbol{E}}_{MN}$, $\tilde{\boldsymbol{H}}_{MN}$ are related to the vectors $\boldsymbol{E}_{MN}$, $\boldsymbol{H}_{MN}$ by (3.17) and (B 27) and

$$\hat{\boldsymbol{D}}_{MN} = a_{33}^{\frac{1}{2}}\,\bar{\boldsymbol{d}}_{MN} = A_{33}^{\frac{1}{2}}\,\bar{\boldsymbol{D}}_{MN}, \quad \hat{\boldsymbol{B}}_{MN} = a_{33}^{\frac{1}{2}}\,\boldsymbol{b}_{MN} = A_{33}^{\frac{1}{2}}\,\boldsymbol{B}_{MN}. \tag{4.2}$$

Then, from (2.18) and (3.16) we have

$$\begin{aligned}-\lambda(\dot{\psi}+\eta\dot{\theta}+\sum\eta_{MN}\,\dot{\theta}_{MN}+\theta\xi+\sum\theta_{MN}\,\xi_{MN})-k\,\partial\theta/\partial\zeta-\sum k_{MN}\,\partial\theta_{MN}/\partial\zeta+P+P_{\mathrm{e}}\\ +\sum(\hat{\boldsymbol{J}}_{MN}\cdot\tilde{\boldsymbol{E}}_{MN}-\hat{\boldsymbol{D}}_{MN}\cdot\tilde{\boldsymbol{E}}_{MN}-\hat{\boldsymbol{B}}_{MN}\cdot\tilde{\boldsymbol{H}}_{MN}) = 0.\end{aligned} \tag{4.3}$$

Once constitutive equations have been specified for

$$\psi,\ \eta,\ \eta_{MN},\ \xi,\ \xi_{MN},\ k,\ k_{MN},\ \hat{\boldsymbol{J}}_{MN},\ \hat{\boldsymbol{D}}_{MN},\ \hat{\boldsymbol{B}}_{MN},\ \boldsymbol{n},\ \boldsymbol{k}^\alpha,\ \boldsymbol{m}^\alpha \tag{4.4}$$

then (4.3) is an identity to be satisfied for all processes subject to the electromagnetic equations (3.6)–(3.9) or (3.11)–(3.14).

A magnetic polarized thermoelastic body is defined to be one for which the variables (4.4) are functions of

$$\theta,\ \theta_{RS},\ \partial\theta/\partial\zeta,\ \partial\theta_{RS}/\partial\zeta,\ \boldsymbol{a}_3,\ \boldsymbol{d}_\alpha,\ \partial\boldsymbol{d}_\alpha/\partial\zeta,\ \tilde{\boldsymbol{E}}_{MN},\ \tilde{\boldsymbol{H}}_{MN}, \tag{4.5}$$

for $R+S = 1, 2, \ldots, P$ and $M+N = 0, 1, \ldots, L$, as well as functions of values in the reference configuration, namely

$$\boldsymbol{\Theta},\ \boldsymbol{D}_\alpha,\ \partial\boldsymbol{D}_\alpha/\partial\zeta,\ \boldsymbol{A}_3. \tag{4.6}$$

In the reference configuration θ_{MN} are zero, $\boldsymbol{\Theta}$ is the constant value of θ, and the electromagnetic variables are absent.

First, we set aside any invariance requirements under superposed rigid body motions, and satisfy the energy identity (4.3) for all processes subject to equations (3.11)–(3.14). It follows that ψ must be independent of θ and $\partial\theta_{RS}/\partial\zeta$. Then, omitting explicit display of the reference variables (4.6) and ζ, we have

$$\psi = \psi_1(\theta, \theta_{RS}, \boldsymbol{a}_3, \boldsymbol{d}_\alpha, \partial\boldsymbol{d}_\alpha/\partial\zeta, \tilde{\boldsymbol{E}}_{MN}, \tilde{\boldsymbol{H}}_{MN}), \tag{4.7a}$$

$$\eta = -\frac{\partial\psi_1}{\partial\theta}, \quad \eta_{RS} = -\frac{\partial\psi}{\partial\theta_{RS}}, \quad \hat{\boldsymbol{D}}_{MN} = -\lambda\frac{\partial\psi_1}{\partial\tilde{\boldsymbol{E}}_{MN}}, \quad \hat{\boldsymbol{B}}_{MN} = -\lambda\frac{\partial\psi_1}{\partial\tilde{\boldsymbol{H}}_{MN}}, \tag{4.7b–e}$$

$$\boldsymbol{n}+\boldsymbol{n}_{\mathrm{e}} = \lambda\frac{\partial\psi_1}{\partial\boldsymbol{a}_3}, \quad \boldsymbol{k}^\alpha+\boldsymbol{k}^\alpha_{\mathrm{e}} = \lambda\frac{\partial\psi_1}{\partial\boldsymbol{d}_\alpha}, \quad \boldsymbol{m}^\alpha+\boldsymbol{m}^\alpha_{\mathrm{e}} = \lambda\frac{\partial\psi_1}{\partial(\partial\boldsymbol{d}_\alpha/\partial\zeta)}. \tag{4.7f–h}$$

Also $$-\lambda(\theta\xi+\Sigma\,\theta_{MN}\xi_{MN})-k\,\partial\theta/\partial\zeta-\Sigma\,k_{MN}\,\partial\theta_{MN}/\partial\zeta+\Sigma\,\hat{\boldsymbol{J}}_{MN}\cdot\tilde{\boldsymbol{E}}_{MN} = 0. \tag{4.8}$$

The expressions (4.7) must satisfy the moment of momentum equation in (2.13). This is equivalent to requiring that ψ must be unaltered under a (static) rigid body rotation. It is, perhaps, simplest to impose this condition on ψ in (4.7) since only $\boldsymbol{a}_3$, $\boldsymbol{d}_\alpha$, $\partial\boldsymbol{d}_\alpha/\partial\zeta$ are changed by a static rigid body rotation and invariance conditions imposed on ψ as a function of these variables have been discussed elsewhere (see, for example, Green & Laws 1966; Green *et al.* 1974; Green & Naghdi 1979). Then, recalling (2.6), and using the notation

$$\left.\begin{aligned} h_{ij} &= \boldsymbol{d}_i\cdot\boldsymbol{d}_j, & d_{\alpha i} &= \boldsymbol{d}_i\cdot\partial\boldsymbol{d}_\alpha/\partial\zeta, & h^{ij} &= \boldsymbol{d}^i\cdot\boldsymbol{d}^j, & d_\alpha{}^i &= \boldsymbol{d}^i\cdot\partial\boldsymbol{d}_\alpha/\partial\zeta,\\ H_{ij} &= \boldsymbol{D}_i\cdot\boldsymbol{D}_j, & D_{\alpha i} &= \boldsymbol{D}_i\cdot\partial\boldsymbol{D}_\alpha/\partial\zeta, & H^{ij} &= \boldsymbol{D}^i\cdot\boldsymbol{D}^j, & D_\alpha{}^i &= \boldsymbol{D}^i\cdot\partial\boldsymbol{D}_\alpha/\partial\zeta,\end{aligned}\right\} \tag{4.9}$$

$$\gamma_{ij} = h_{ij}-H_{ij}, \quad \kappa_{\alpha i} = d_{\alpha i}-D_{\alpha i},$$

we see that ψ may be replaced by

$$\psi = \psi_2(\theta, \theta_{RS}, \gamma_{ij}, \kappa_{\alpha i}, \tilde{\boldsymbol{E}}_{MN}, \tilde{\boldsymbol{H}}_{MN}) \tag{4.10}$$

if we omit the reference variables. Constitutive equations in component forms then follow as in previous papers, with the additional results here concerning electromagnetic variables. For our purpose we record the results in a form similar to that used by Green & Laws (1966). If

$$\left.\begin{aligned} \boldsymbol{n} &= n^i\boldsymbol{d}_i, & \boldsymbol{k}^\alpha &= k^{\alpha i}\boldsymbol{d}_i, & \boldsymbol{m}^\alpha &= m^{\alpha i}\boldsymbol{d}_i,\\ \boldsymbol{n}_{\mathrm{e}} &= n^i_{\mathrm{e}}\boldsymbol{d}_i, & \boldsymbol{k}^\alpha_{\mathrm{e}} &= k^{\alpha i}_{\mathrm{e}}\boldsymbol{d}_i, & \boldsymbol{m}^\alpha_{\mathrm{e}} &= m^{\alpha i}_{\mathrm{e}}\boldsymbol{d}_i,\end{aligned}\right\} \tag{4.11}$$

then

$$\left.\begin{aligned} n^3+n_e^3-d_\alpha^{\cdot 3}(m^{\alpha 3}+m_e^{\alpha 3}) = 2\lambda\frac{\partial\psi_2}{\partial\gamma_{33}},\quad n^\alpha+n_e^\alpha-d_\beta^{\cdot\alpha}(m^{\beta 3}+m_e^{\beta 3}) = \lambda\frac{\partial\psi_2}{\partial\gamma_{\alpha 3}},\\ k^{\alpha\beta}+k^{\beta\alpha}+k_e^{\alpha\beta}+k_e^{\beta\alpha}-d_\nu^{\cdot\alpha}(m^{\nu\beta}+m_e^{\nu\beta})-d_\nu^{\cdot\beta}(m^{\nu\alpha}+m_e^{\nu\alpha}) = 4\lambda\frac{\partial\psi_2}{\partial\gamma_{\alpha\beta}},\\ m^{\alpha i}+m_e^{\alpha i} = \lambda\frac{\partial\psi_2}{\partial\kappa_{\alpha i}},\quad \eta = -\frac{\partial\psi_2}{\partial\theta},\quad \eta_{RS} = -\frac{\partial\psi_2}{\partial\theta_{RS}},\\ \hat{D}_{MN}{}^i = -\lambda\frac{\partial\psi_2}{\partial\tilde{E}_{MNi}},\quad B_{MN}{}^i = -\lambda\frac{\partial\psi_2}{\partial\tilde{H}_{MNi}}. \end{aligned}\right\} \tag{4.12}$$

In evaluating (4.12), ψ_2 is regarded as a function of γ_{33}, $\gamma_{\alpha 3}$, $\frac{1}{2}(\gamma_{\alpha\beta}+\gamma_{\beta\alpha})$, $\tilde{E}_{MNi}$ and $\tilde{H}_{MNi}$.

5. Linear theory for straight rods

We consider here the linear theory of a thermoelastic magnetic polarized Cosserat curve, which corresponds to the theory of a straight rod of constant cross-section. We suppose that the rod is unstressed and at uniform temperature $\bar{\theta}$, constant density ρ^* and without electromagnetic fields in its reference configuration. The linearization procedure and resulting equations are already available for a thermoelastic Cosserat curve. Extension of this to include linearization of the electromagnetic aspects of the theory follows similar lines so we omit details and only record the final results.

We choose the reference line and directors to be given by

$$\boldsymbol{R} = \zeta\boldsymbol{e}_3,\quad \boldsymbol{D}_\alpha = \boldsymbol{e}_\alpha,\quad \boldsymbol{D}_3 = \boldsymbol{e}_3,\quad \boldsymbol{D}^i = \boldsymbol{e}_i,\quad A_{33} = 1, \tag{5.1 a–e}$$

where $\boldsymbol{e}_i$ is a constant orthonormal system of vectors. All equations will refer to the reference configuration represented by (5.1 *a–c*). The motion of the Cosserat rod as given by (2.1), (2.3) and (2.5) is now specified by

$$\left.\begin{aligned} \boldsymbol{r} = \boldsymbol{R}+\boldsymbol{u},\quad \boldsymbol{d}_i = \boldsymbol{D}_i+\boldsymbol{\delta}_i,\quad \boldsymbol{v} = \dot{\boldsymbol{u}},\quad \boldsymbol{w}_\alpha = \dot{\boldsymbol{\delta}}_\alpha,\\ \boldsymbol{u} = u_i\boldsymbol{e}_i,\quad \boldsymbol{\delta}_i = \bar{\delta}_{ij}\boldsymbol{e}_j. \end{aligned}\right\} \tag{5.2}$$

Linear kinematic measures of deformation based on (5.2) are

$$\gamma_{ij} = \boldsymbol{e}_i\cdot\boldsymbol{\delta}_j+\boldsymbol{e}_j\cdot\boldsymbol{\delta}_i = \bar{\delta}_{ij}+\bar{\delta}_{ji},\quad \kappa_{\alpha i} = \partial\bar{\delta}_{\alpha i}/\partial\zeta,\quad \bar{\delta}_{3i} = \partial u_i/\partial\zeta. \tag{5.3}$$

Within the order of approximation of the linear theory, from §2 and equations (5.2) we have

$$\boldsymbol{\nu} = {}_{\mathrm{R}}\boldsymbol{\nu} = \boldsymbol{e}_3,\quad \boldsymbol{k}^\alpha = \boldsymbol{k}^{\alpha 3} = {}_{\mathrm{R}}\boldsymbol{k}^{\alpha 3},\quad \lambda = \rho = {}_{\mathrm{R}}\rho. \tag{5.4}$$

For convenience, in the rest of this section we omit the subscript R but understand that all response functions are defined with respect to the reference configuration. Then, from (2.13), the equations of motion are

$$\left.\begin{aligned} \rho(\ddot{\boldsymbol{u}}+y^{0\beta}\ddot{\boldsymbol{\delta}}_\beta) = \rho\boldsymbol{f}+\partial\boldsymbol{n}/\partial\zeta,\\ \rho(y^{0\alpha}\ddot{\boldsymbol{u}}+y^{\alpha\beta}\ddot{\boldsymbol{\delta}}_\beta) = \rho\boldsymbol{l}^\alpha-\boldsymbol{k}^\alpha+\partial\boldsymbol{m}^\alpha/\partial\zeta,\\ \boldsymbol{e}_3\times\boldsymbol{n}+\boldsymbol{e}_\alpha\times\boldsymbol{k}^\alpha = \boldsymbol{0}, \end{aligned}\right\} \tag{5.5}$$

where second-order terms due to the electromagnetic fields are omitted and where

$$\boldsymbol{n} = n_i \boldsymbol{e}_i, \quad \boldsymbol{k}^\alpha = k_{\alpha i} \boldsymbol{e}_i, \quad \boldsymbol{m}^\alpha = m_{\alpha i} \boldsymbol{e}_i, \quad \boldsymbol{f} = f_i \boldsymbol{e}_i, \quad \boldsymbol{l}^\alpha = l_{\alpha i} \boldsymbol{e}_i. \tag{5.6}$$

The component forms of (5.5) are then

$$\rho(\ddot{u}_i + y^{0\beta} \ddot{\delta}_{\beta i}) = \rho f_i + \partial n_i/\partial\zeta, \tag{5.7a}$$

$$\rho(y^{0\alpha} \ddot{u}_i + y^{\alpha\beta} \ddot{\delta}_{\beta i}) = \rho l_{\alpha i} - k_{\alpha i} + \partial m_{\alpha i}/\partial\zeta, \tag{5.7b}$$

$$n_\alpha = k_{\alpha 3}, \quad k_{12} = k_{21}. \tag{5.7c, d}$$

We restrict attention to three temperatures and replace θ by $\bar{\theta} + \theta$ where θ, $\theta_{10} = \theta_1$, and $\theta_{01} = \theta_2$ are small compared with $\bar{\theta}$. The field equations of entropy balance are then

$$\left.\begin{aligned} &\rho\dot{\eta} = \rho(s+\xi) - \partial k/\partial\zeta, \quad \rho\dot{\eta}_\alpha = \rho(s_\alpha + \xi_\alpha) - \partial k_\alpha/\partial\zeta, \\ &h = \bar{\theta} k, \quad r = \bar{\theta} s, \quad h_{10} = h_{01} = 0, \quad r_{10} = r_{01} = 0, \\ &\eta_1 = \eta_{10}, \quad \eta_2 = \eta_{01}, \quad s_1 = s_{10}, \quad s_2 = s_{01}, \quad \xi_1 = \xi_{10}, \quad \xi_2 = \xi_{01}, \\ &k_1 = k_{10}, \quad k_2 = k_{01}. \end{aligned}\right\} \tag{5.8}$$

Turning to the electromagnetic fields, in view of linearization we have

$$\left.\begin{aligned} &\boldsymbol{E}_{MN} = \boldsymbol{e}^*_{MN} = E_{MNi} \boldsymbol{e}_i, \quad \boldsymbol{H}_{MN} = \boldsymbol{h}^*_{MN} = H_{MNi} \boldsymbol{e}_i, \\ &\boldsymbol{B}_{MN} = \boldsymbol{b}_{MN} = B_{MNi} \boldsymbol{e}_i, \quad \bar{\boldsymbol{D}}_{MN} = \bar{\boldsymbol{d}}_{MN} = \bar{D}_{MNi} \boldsymbol{e}_i, \\ &\boldsymbol{J}_{MN} = \boldsymbol{j}^*_{MN} = J_{MNi} \boldsymbol{e}_i, \quad E_{MN} = e_{MN} \end{aligned}\right\} \tag{5.9}$$

and from (3.11)–(3.14) we see that the field equations reduce to

$$-B'_{MN} + \sum_{K=0}^{M} \chi^M_K B_{KN1} + \sum_{K=0}^{N} \bar{\chi}^N_K B_{MK2} = \partial B_{MN3}/\partial\zeta, \tag{5.10}$$

$$-D'_{MN} + E_{MN} + \sum_{K=0}^{M} \psi^M_K \bar{D}_{KN1} + \sum_{K=0}^{N} \bar{\psi}^N_K \bar{D}_{MK2} = \partial \bar{D}_{MN3}/\partial\zeta, \tag{5.11}$$

$$\left.\begin{aligned} &\dot{B}_{MN\alpha} = -e_{\beta\alpha} \partial E_{MN\beta}/\partial\zeta + \boldsymbol{e}_\alpha \cdot \boldsymbol{E}'_{MN} + e_{\alpha 1} \sum_{K=0}^{M} \chi^M_K E_{KN3} + e_{\alpha 2} \sum_{K=0}^{N} \bar{\chi}^N_K E_{MK3}, \\ &\dot{B}_{MN3} = \boldsymbol{e}_3 \cdot \boldsymbol{E}'_{MN} + \sum_{K=0}^{M} \chi^M_K E_{KN2} - \sum_{K=0}^{N} \bar{\chi}^N_K E_{MK1}, \end{aligned}\right\} \tag{5.12}$$

$$\left.\begin{aligned} &-\dot{\bar{D}}_{MN\alpha} = J_{MN\alpha} - e_{\beta\alpha} \partial H_{MN\beta}/\partial\zeta + \boldsymbol{e}_\alpha \cdot \boldsymbol{H}'_{MN} + e_{\alpha 1} \sum_{K=0}^{M} \psi^M_K H_{KN3} + e_{\alpha 2} \sum_{K=0}^{N} \bar{\psi}^N_K H_{MK3}, \\ &-\dot{\bar{D}}_{MN3} = J_{MN3} + \boldsymbol{e}_3 \cdot \boldsymbol{H}'_{MN} + \sum_{K=0}^{M} \psi^M_K H_{KN2} - \sum_{K=0}^{N} \bar{\psi}^N_K H_{MK1}, \end{aligned}\right\} \tag{5.13}$$

where B'_{MN}, D'_{MN}, $\boldsymbol{H}'_{MN}$, $\boldsymbol{E}'_{MN}$ are given in terms of surface values of the electromagnetic fields by (B 22)–(B 24) and $e_{11} = e_{22} = 0$, $e_{12} = -e_{21} = 1$.

Finally, from (4.3), the energy equation reduces to

$$\begin{aligned} &-\rho\{\dot{\psi} + \eta\dot{\theta} + \eta_\alpha \dot{\theta}_\alpha + \xi(\bar{\theta} + \theta) + \theta_\alpha \xi_\alpha\} - k\,\partial\theta/\partial\zeta - k_\alpha\,\partial\theta_\alpha/\partial\zeta \\ &\qquad + \Sigma\,(\hat{\boldsymbol{J}}_{MN} \cdot \tilde{\boldsymbol{E}}_{MN} - \bar{\boldsymbol{D}}_{MN} \cdot \dot{\tilde{\boldsymbol{E}}}_{MN} - \boldsymbol{B}_{MN} \dot{\tilde{\boldsymbol{H}}}_{MN}) \\ &\qquad\qquad + \boldsymbol{n} \cdot \partial\dot{\boldsymbol{u}}/\partial\zeta + \boldsymbol{k}^\alpha \cdot \dot{\boldsymbol{\delta}}_\alpha + \boldsymbol{m}^\alpha \cdot \partial\dot{\boldsymbol{\delta}}_\alpha/\partial\zeta = 0. \end{aligned} \tag{5.14}$$

Then, for a magnetic polarized thermoelastic rod in the linearized theory we have, either from (5.14) and (5.7 c, d) or from the results in (4.12),

$$\left.\begin{aligned} &n_3 = 2\rho\frac{\partial\psi}{\partial\gamma_{33}}, \quad n_\alpha = k_{\alpha3} = \rho\frac{\partial\psi}{\partial\gamma_{\alpha3}}, \quad k_{\alpha\beta} = k_{\beta\alpha} = 2\rho\frac{\partial\psi}{\partial\gamma_{\alpha\beta}}, \\ &m_{\alpha i} = \rho\frac{\partial\psi}{\partial\kappa_{\alpha i}}, \quad \eta = -\frac{\partial\psi}{\partial\theta}, \quad \eta_\alpha = -\frac{\partial\psi}{\partial\theta_\alpha}, \\ &\bar{D}_{MNi} = -\rho\frac{\partial\psi}{\partial\tilde{E}_{MNi}}, \quad B_{MNi} = -\rho\frac{\partial\psi}{\partial\tilde{H}_{MNi}}, \\ &\psi = \psi(\gamma_{33}, \gamma_{\alpha3}, \tfrac{1}{2}(\gamma_{\alpha\beta}+\gamma_{\beta\alpha}), \kappa_{\alpha i}, \theta, \theta_\alpha, \tilde{E}_{MNi}, \tilde{H}_{MNi}), \\ &-\rho\{(\bar{\theta}+\theta)\,\xi+\theta_\alpha\,\xi_\alpha\} - k\,\partial\theta/\partial\zeta - k_\alpha\,\partial\theta_\alpha/\partial\zeta + \textstyle\sum \boldsymbol{J}_{MN}\cdot\tilde{\boldsymbol{E}}_{MN} = 0. \end{aligned}\right\} \tag{5.15}$$

In the rest of this section we limit our attention to the case where $M+N = 0, 1$, i.e. the electromagnetic vectors are limited to $\boldsymbol{E}_{00}$, $\boldsymbol{E}_{10}$, $\boldsymbol{E}_{01}$, $\boldsymbol{H}_{00}$, $\boldsymbol{H}_{10}$, $\boldsymbol{H}_{01}$, $\bar{\boldsymbol{D}}_{00}$, $\bar{\boldsymbol{D}}_{10}$, $\bar{\boldsymbol{D}}_{01}$, $\boldsymbol{B}_{00}$, $\boldsymbol{B}_{10}$, $\boldsymbol{B}_{01}$, $\boldsymbol{J}_{00}$, $\boldsymbol{J}_{10}$, $\boldsymbol{J}_{01}$ with corresponding vectors $\tilde{\boldsymbol{E}}_{00}$, $\tilde{\boldsymbol{E}}_{10}$, $\tilde{\boldsymbol{E}}_{01}$, $\tilde{\boldsymbol{H}}_{00}$, $\tilde{\boldsymbol{H}}_{10}$, $\tilde{\boldsymbol{H}}_{01}$, where, from (3.18),

$$E_{MNi} = \sum_{R+S=0}^{1} I_{(i)}^{MNRS}\,\tilde{E}_{RSi}, \quad H_{MNi} = \sum_{R+S=0}^{1} K_{(i)}^{MNRS}\,\tilde{H}_{RSi} \quad (M+N = 0, 1) \tag{5.16}$$

with coefficients given by (B 28).

Since the rod in its reference configuration is at constant temperature, homogeneous, of constant density, and without electromagnetic fields, ψ is a quadratic function of the variables in (5.15) with constant coefficients. To proceed with the identification of constitutive coefficients, it is more convenient to express (5.15) in a partly inverted form. For this purpose, we introduce the Gibbs free energy function G by

$$\left.\begin{aligned} \rho G &= \rho\psi - \tfrac{1}{2}n_3\,\gamma_{33} - n_\alpha\,\gamma_{\alpha3} - \tfrac{1}{2}k_{\alpha\beta}\,\gamma_{\alpha\beta} - m_{\alpha i}\,\kappa_{\alpha i}, \\ G &= G(n_3, n_\alpha, k_{\alpha\beta}, m_{\alpha i}, \theta, \theta_\alpha, \tilde{E}_{MNi}, \tilde{H}_{MNi}). \end{aligned}\right\} \tag{5.17}$$

where

Then,

$$\left.\begin{aligned} &\gamma_{33} = -2\rho\frac{\partial G}{\partial n_3}, \quad \gamma_{\alpha3} = -\rho\frac{\partial G}{\partial n_\alpha}, \quad \gamma_{\alpha\beta} = -\rho\left(\frac{\partial G}{\partial k_{\alpha\beta}}+\frac{\partial G}{\partial k_{\beta\alpha}}\right), \\ &\eta = -\frac{\partial G}{\partial\theta}, \quad \eta_\alpha = -\frac{\partial G}{\partial\theta_\alpha}, \quad \bar{D}_{MNi} = -\rho\frac{\partial G}{\partial\tilde{E}_{MNi}}, \quad B_{MNi} = -\rho\frac{\partial G}{\partial\tilde{H}_{MNi}}, \\ &\kappa_{\alpha i} = -\rho\frac{\partial G}{\partial m_{\alpha i}}. \end{aligned}\right\} \tag{5.18}$$

For the linearized theory under discussion, G is a quadratic form given by

$$\begin{aligned} \rho G = &-\tfrac{1}{2}A_{3333}\,n_3^2 - 2A_{33\alpha3}\,n_3\,n_\alpha - A_{33\alpha\beta}\,n_3\,k_{\alpha\beta} - 2A_{\alpha3\beta3}\,n_\alpha\,n_\beta \\ &- 2A_{\alpha3\lambda\mu}\,n_\alpha\,k_{\lambda\mu} - \tfrac{1}{2}A_{\alpha\beta\lambda\mu}\,k_{\alpha\beta}\,k_{\lambda\mu} - A^{\alpha}_{33k}\,n_3\,m_{\alpha k} - 2A^{\alpha}_{\lambda3k}\,n_\lambda\,m_{\alpha k} - A^{\alpha}_{\lambda\mu k}\,k_{\lambda\mu}\,m_{\alpha k} \\ &+ \textstyle\sum(C^{MN}_{33k}\,n_3 + 2C^{MN}_{\alpha3k}\,n_\alpha + C^{MN}_{\alpha\beta k}\,k_{\alpha\beta})\,\tilde{E}_{MNk} + \sum(F^{MN}_{33k}\,n_3 + 2F^{MN}_{\alpha3k}\,n_\alpha + F^{MN}_{\alpha\beta k}\,k_{\alpha\beta})\,\tilde{H}_{MNk} \\ &- P_{33}\,n_3\,\theta - 2P_{\alpha3}\,n_\alpha\,\theta - P_{\alpha\beta}\,k_{\alpha\beta}\,\theta - P^{\alpha}_{33}\,n_3\,\theta_\alpha - 2P^{\alpha}_{\lambda3}\,n_\lambda\,\theta_\alpha - P^{\alpha}_{\lambda\mu}\,k_{\lambda\mu}\,\theta_\alpha \\ &- \tfrac{1}{2}B^{\alpha\beta}_{ij}\,m_{\alpha i}\,m_{\beta j} + \textstyle\sum C^{\alpha MN}_{ij}\,m_{\alpha i}\,\tilde{E}_{MNj} + \sum F^{\alpha MN}_{ij}\,m_{\alpha i}\,\tilde{H}_{MNj} \end{aligned}$$

$$-P_i^{\alpha 0} m_{\alpha i}\theta - P_i^{\alpha\beta} m_{\alpha i}\theta_\beta - \tfrac{1}{2}P\theta^2 - P^{\alpha 0}\theta_\alpha\theta - \tfrac{1}{2}\bar{P}^{\alpha\beta}\theta_\alpha\theta_\beta$$

$$+\Sigma(\tfrac{1}{2}L_{ij}^{MNRS}\tilde{E}_{MNi}\tilde{E}_{RSj} + M_{ij}^{MNRS}\tilde{E}_{MNi}\tilde{H}_{RSj} + \tfrac{1}{2}N_{ij}^{MNRS}\tilde{H}_{MNi}\tilde{H}_{RSj})$$

$$+\Sigma(R_i^{MN}\tilde{E}_{MNi}\theta + R_i^{MN\alpha}\tilde{E}_{MNi}\theta_\alpha + S_i^{MN}\tilde{H}_{MNi}\theta + S_i^{MN\alpha}\tilde{H}_{MNi}\theta_\alpha) \tag{5.19}$$

for $M+N = 0, 1$, $R+S = 0, 1$. All the coefficients in (5.17) are constants and they possess the following symmetries:

$$\left.\begin{aligned}
A_{33\alpha\beta} = A_{33\beta\alpha},\quad A_{\alpha 3\lambda\mu} = A_{\alpha 3\mu\lambda},\quad A_{\alpha 3\beta 3} = A_{\beta 3\alpha 3},\\
A_{\alpha\beta\lambda\mu} = A_{\alpha\beta\mu\lambda} = A_{\beta\alpha\lambda\mu} = A_{\lambda\mu\alpha\beta},\quad A_{\lambda\mu k}^{\alpha} = A_{\mu\lambda k}^{\alpha},\\
C_{\alpha\beta k}^{MN} = C_{\beta\alpha k}^{MN},\quad F_{\alpha\beta k}^{MN} = F_{\beta\alpha k}^{MN},\quad P_{\alpha\beta} = P_{\beta\alpha},\quad P_{\lambda\mu}^{\alpha} = P_{\mu\lambda}^{\alpha},\\
B_{ij}^{\alpha\beta} = B_{ji}^{\beta\alpha},\quad \bar{P}^{\alpha\beta} = \bar{P}^{\beta\alpha},\quad L_{ij}^{MNRS} = L_{ji}^{RSMN},\quad N_{ij}^{MNRS} = N_{ji}^{RSMN}.
\end{aligned}\right\} \tag{5.20}$$

From (5.18) and (5.19) we then obtain the constitutive relations

$$\left.\begin{aligned}
\tfrac{1}{2}\gamma_{33} &= A_{3333}n_3 + 2A_{33\alpha 3}n_\alpha + A_{33\alpha\beta}k_{\alpha\beta} + A_{33k}^{\alpha}m_{\alpha k}\\
&\quad -\Sigma\,(C_{33k}^{MN}\tilde{E}_{MNk} + F_{33k}^{MN}\tilde{H}_{MNk}) + P_{33}\theta + P_{33}^{\alpha}\theta_\alpha,\\
\tfrac{1}{2}\gamma_{\alpha 3} &= A_{33\alpha 3}n_3 + 2A_{\alpha 3\beta 3}n_\beta + A_{\alpha 3\lambda\mu}k_{\lambda\mu} + A_{\alpha 3k}^{\beta}m_{\beta k}\\
&\quad -\Sigma\,(C_{\alpha 3k}^{MN}\tilde{E}_{MNk} + F_{\alpha 3k}^{MN}\tilde{H}_{MNk}) + P_{\alpha 3}\theta + P_{\alpha 3}^{\beta}\theta_\beta,\\
\tfrac{1}{2}\gamma_{\alpha\beta} &= A_{33\alpha\beta}n_3 + 2A_{\lambda 3\alpha\beta}n_\lambda + A_{\alpha\beta\lambda\mu}k_{\lambda\mu} + A_{\alpha\beta k}^{\lambda}m_{\lambda k}\\
&\quad -\Sigma\,(C_{\alpha\beta k}^{MN}\tilde{E}_{MNk} + F_{\alpha\beta k}^{MN}\tilde{H}_{MNk}) + P_{\alpha\beta}\theta + P_{\alpha\beta}^{\lambda}\theta_\lambda,\\
\rho\eta &= P_{33}n_3 + 2P_{\alpha 3}n_\alpha + P_{\alpha\beta}k_{\alpha\beta} + P_i^{\alpha 0}m_{\alpha i} + P\theta + P^{\alpha 0}\theta_\alpha\\
&\quad -\Sigma\,(R_i^{MN}\tilde{E}_{MNi} + S_i^{MN}\tilde{H}_{MNi}),\\
\bar{D}_{MNi} &= -C_{33i}^{MN}n_3 - 2C_{\alpha 3i}^{MN}n_\alpha - C_{\alpha\beta i}^{MN}k_{\alpha\beta} - C_{ji}^{\beta MN}m_{\beta j}\\
&\quad -\sum_{R,S}(L_{ij}^{MNRS}\tilde{E}_{RSj} + M_{ij}^{MNRS}\tilde{H}_{RSj}) - R_i^{MN}\theta - R_i^{MN\alpha}\theta_\alpha,\\
\kappa_{\alpha i} &= B_{ij}^{\alpha\beta}m_{\beta j} + A_{33i}^{\alpha}n_3 + 2A_{\lambda 3i}^{\alpha}n_\lambda + A_{\lambda\mu i}^{\alpha}k_{\lambda\mu}\\
&\quad -\Sigma\,(C_{ij}^{\alpha MN}\tilde{E}_{MNj} + F_{ij}^{\alpha MN}\tilde{H}_{MNj}) + P_i^{\alpha 0}\theta + P_i^{\alpha\beta}\theta_\beta,\\
\rho\eta_\alpha &= P_{33}^{\alpha}n_3 + 2P_{\lambda 3}^{\alpha}n_\lambda + P_{\lambda\mu}^{\alpha}k_{\lambda\mu} + P_i^{\beta\alpha}m_{\beta i} + P^{\alpha 0}\theta + \bar{P}^{\alpha\beta}\theta_\beta\\
&\quad -\Sigma\,(R_i^{MN\alpha}\tilde{E}_{MNi} + S_i^{MN\alpha}\tilde{H}_{MNi}),\\
B_{MNi} &= -F_{33i}^{MN}n_3 - 2F_{\alpha 3i}^{MN}n_\alpha - F_{\alpha\beta i}^{MN}k_{\alpha\beta} - F_{ji}^{\beta MN}m_{\beta j}\\
&\quad -\sum_{R,S}(M_{ji}^{RSMN}\tilde{E}_{RSj} + N_{ij}^{MNRS}\tilde{H}_{RSj}) - S_i^{MN}\theta - S_i^{MN\alpha}\theta_\alpha.
\end{aligned}\right\} \tag{5.21}$$

We now further restrict attention to a Cosserat curve which models a straight three-dimensional rod, with constant cross-sections, and which has the line of centroids along the ζ-axis, and geometrical axes of symmetry in each section with respect to directions $\boldsymbol{e}_\alpha$. Values of constitutive coefficients in (5.21) are then specified with the help of comparisons between solutions of the equations of the Cosserat curve with corresponding solutions of the three-dimensional equations. The values will depend on which three-dimensional representation is chosen in Appendix B for the various thermomechanical and electromagnetic vectors, i.e. which weighting functions are used to select λ^α, μ^M, $\bar{\mu}^N$, χ^M, $\bar{\chi}^N$, ψ^M, $\bar{\psi}^N$. The appropriate choice for these functions depends to some extent on the type of boundary conditions imposed on the major surface of the rod, especially as far as the electromagnetic variables are concerned, and on the geometric shape of the cross-sections. Here we consider only the choice of functions

$$\left.\begin{aligned}&\lambda^\alpha(\zeta^1,\zeta^2)=\zeta^\alpha,\quad \mu^1(\zeta^1)=\zeta^1,\quad \bar{\mu}^1(\zeta^2)=\zeta^2,\quad \chi^0(\zeta^1)=1,\quad \chi^1(\zeta^1)=\zeta^1,\\&\psi^0(\zeta^1)=1,\quad \psi^1(\zeta^1)=\zeta^1,\quad \bar{\chi}^0(\zeta^2)=1,\quad \bar{\chi}^1(\zeta^2)=\zeta^2,\quad \bar{\psi}^0(\zeta^2)=1,\quad \bar{\psi}^1(\zeta^2)=\zeta^2,\\&\chi^0_0=\psi^0_0=\bar{\chi}^0_0=\bar{\psi}^0_0=0,\quad \chi^1_0=\psi^1_0=\bar{\chi}^1_0=\bar{\psi}^1_0=1,\quad \chi^1_1=\psi^1_1=\bar{\chi}^1_1=\bar{\psi}^1_1=0,\\&\chi^0_1=\bar{\chi}^0_1=\psi^0_1=\bar{\psi}^0_1=0.\end{aligned}\right\}\tag{5.22}$$

In view of the geometrical symmetry of the cross-sections it follows from (5.16), (5.22) and (B 28) that

$$\left.\begin{aligned}E_{00i}=A\tilde{E}_{00i},\quad E_{10i}=I_{11}\tilde{E}_{10i},\quad E_{01i}=I_{22}\tilde{E}_{01i},\\H_{00i}=A\tilde{H}_{00i},\quad H_{10i}=I_{11}\tilde{H}_{10i},\quad H_{01i}=I_{22}\tilde{H}_{01i},\end{aligned}\right\}\tag{5.23}$$

where
$$A=\iint \mathrm{d}A,\quad I_{11}=\iint \zeta^1\zeta^1\,\mathrm{d}A,\quad I_{22}=\iint\zeta^2\zeta^2\,\mathrm{d}A.\tag{5.24}$$

With the help of Appendix C, constitutive coefficients are chosen to have the values:

$$\left.\begin{aligned}&A^\alpha_{ijk}=0,\quad P^\alpha_{ij}=0,\quad P^{\alpha0}_i=0,\quad P^{\alpha0}=0,\\&A_{ijrs}=A^{-1}s_{ijrs}\\&\text{except for}\\&A_{\alpha3\beta3}=\beta_{(\alpha\beta)}A^{-1}s_{\alpha3\beta3},\quad A_{\alpha312}=\beta_{(\alpha)}A^{-1}s_{\alpha312},\quad A_{1212}=\beta A^{-1}s_{1212},\\&P_{ij}=s_{ij},\quad P=Ac^*,\quad \bar{P}^{11}=I_{11}c^*,\quad \bar{P}^{22}=I_{22}c^*,\quad \bar{P}^{12}=0,\\&P^{11}_\lambda=P^{22}_\lambda=2s_{\lambda3},\quad P^{11}_3=P^{22}_3=s_{33},\quad P^{21}_i=0,\quad P^{12}_i=0,\\&B^{11}_{\alpha3}=B^{11}_{3\alpha}=2s_{\alpha333}/I_{11},\quad B^{22}_{\alpha3}=B^{22}_{3\alpha}=2s_{\alpha333}/I_{22},\\&B^{11}_{33}=s_{3333}/I_{11},\quad B^{22}_{33}=s_{3333}/I_{22},\quad B^{11}_{22}=B^{22}_{11}=2/\mathscr{D},\\&B^{11}_{1\alpha}=B^{11}_{\alpha1}=4s_{13\alpha3}/I_{11},\quad B^{22}_{2\alpha}=B^{22}_{\alpha2}=4s_{23\alpha3}/I_{22},\\&B^{12}_{1\alpha}=B^{21}_{\alpha1}=0,\quad B^{12}_{2\alpha}=B^{21}_{\alpha2}=0,\quad B^{12}_{13}=B^{21}_{31}=0,\\&B^{12}_{23}=B^{21}_{32}=0,\quad B^{12}_{33}=B^{21}_{33}=0,\quad B^{21}_{13}=B^{12}_{31}=0,\quad B^{21}_{23}=B^{12}_{32}=0,\end{aligned}\right\}\tag{5.25a}$$

and

$$\left.\begin{aligned}&C^{00}_{ijk}=k^*_{ijk},\quad C^{\beta00}_{ij}=0,\quad C^{110}_{\alpha i}=C^{201}_{\alpha i}=2k^*_{\alpha3i},\\&C^{110}_{3i}=C^{201}_{3i}=k^*_{33i},\quad C^{10}_{ijk}=0,\quad C^{01}_{ijk}=0,\quad C^{101}_{ij}=0,\quad C^{210}_{ij}=0,\\&F^{00}_{ijk}=l^*_{ijk},\quad F^{\beta00}_{ij}=0,\quad F^{110}_{\alpha i}=F^{201}_{\alpha i}=2l^*_{\alpha3i},\end{aligned}\right\}\tag{5.25b}$$

$$\left.\begin{aligned} &F^{110}_{3i} = F^{201}_{3i} = l^*_{33i}, \quad F^{10}_{ijk} = 0, \quad F^{01}_{ijk} = 0, \quad F^{101}_{ij} = 0, \quad F^{210}_{ij} = 0, \\ &L^{0000}_{ij} = Af^*_{ij}, \quad L^{1010}_{ij} = I_{11} f^*_{ij}, \quad L^{0101}_{ij} = I_{22} f^*_{ij}, \\ &L^{0010}_{ij} = 0, \quad L^{0001}_{ij} = 0, \quad L^{1000}_{ij} = 0, \quad L^{1001}_{ij} = 0, \quad L^{0100}_{ij} = 0, \quad L^{0110}_{ij} = 0, \end{aligned}\right\} \quad (5.25\,b, \textit{cont.})$$

and

$$\left.\begin{aligned} &M^{0000}_{ij} = Ah^*_{ij}, \quad M^{1010}_{ij} = I_{11} h^*_{ij}, \quad M^{0101}_{ij} = I_{22} h^*_{ij}, \\ &M^{0010}_{ij} = 0, \quad M^{0001}_{ij} = 0, \quad M^{1000}_{ij} = 0, \quad M^{1001}_{ij} = 0, \quad M^{0100}_{ij} = 0, \quad M^{0110}_{ij} = 0, \\ &N^{0000}_{ij} = Ag^*_{ij}, \quad N^{1010}_{ij} = I_{11} g^*_{ij}, \quad N^{0101}_{ij} = I_{22} g^*_{ij}, \\ &N^{0010}_{ij} = 0, \quad N^{0001}_{ij} = 0, \quad N^{1000}_{ij} = 0, \quad N^{1001}_{ij} = 0, \quad N^{0100}_{ij} = 0, \quad N^{0110}_{ij} = 0, \\ &R^{00}_{i} = Af^*_{i}, \quad R^{10}_{i} = 0, \quad R^{01}_{i} = 0, \quad R^{00\alpha}_{i} = 0, \\ &R^{101}_{i} = I_{11} f^*_{i}, \quad R^{012}_{i} = I_{22} f^*_{i}, \quad R^{102}_{i} = 0, \quad R^{011}_{i} = 0, \\ &S^{00}_{i} = Ag^*_{i}, \quad S^{10}_{i} = 0, \quad S^{01}_{i} = 0, \quad S^{00\alpha}_{i} = 0, \\ &S^{101}_{i} = I_{11} g^*_{i}, \quad S^{012}_{i} = I_{22} g^*_{i}, \quad S^{102}_{i} = 0, \quad S^{001}_{i} = 0. \end{aligned}\right\} \quad (5.25\,c)$$

In (5.25) $\mathscr{D}$ is the torsional rigidity of the rod and $\beta_{(\alpha\beta)}$, $\beta_{(\alpha)}$ are coefficients to be specified.

When discussing constitutive equations for the response functions k, k_α, ξ, ξ_α and J_{MN}, by the direct method we need restrictions which may arise from further thermodynamical considerations. The constitutive coefficients in these equations are then expressed in terms of the three-dimensional coefficients in (C 1) with the help of results in Appendix B. Here, however, we make direct use of (C 2) and Appendix B, together with (5.22), and list the final results:

$$\left.\begin{aligned} &k = -Ak_{33}\,\partial\theta/\partial\zeta - Ak_{3\alpha}\,\theta_\alpha - \bar{a}_{3i}\,A\tilde{E}_{00i}, \\ &k_1 = -I_{11}(k_{33}\,\partial\theta_1/\partial\zeta + \bar{a}_{3i}\,\tilde{E}_{10i}), \quad k_2 = -I_{22}(k_{22}\,\partial\theta_2/\partial\zeta + \bar{a}_{3i}\,\tilde{E}_{01i}), \\ &\xi = 0, \\ &\rho\xi_1 = -Ak_{13}\,\partial\theta/\partial\zeta - Ak_{1\beta}\,\theta_\beta - A\bar{a}_{1i}\,\tilde{E}_{00i}, \quad \rho\xi_2 = -Ak_{23}\,\partial\theta/\partial\zeta - Ak_{2\beta}\,\theta_\beta - A\bar{a}_{2i}\,\tilde{E}_{00i}, \\ &J^{00}_i = Al_{i3}\,\partial\theta/\partial\zeta + Al_{i\alpha}\,\theta_\alpha + Ab_{ij}\,\tilde{E}_{00j}, \\ &J^{10}_i = I_{11}(l_{i3}\,\partial\theta_1/\partial\zeta + b_{ij}\,\tilde{E}_{10j}), \quad J^{01}_i = I_{22}(l_{i3}\,\partial\theta_2/\partial\zeta + b_{ij}\,\tilde{E}_{01j}). \end{aligned}\right\} \quad (5.26)$$

6. A restricted theory of rods

For many purposes it is sufficient to develop a theory of rods which has a simpler structure than that of §2. This theory may be regarded as a constrained version of the theory given in §2 or may be obtained as a direct theory *ab initio* by separate postulates. We adopt the latter course and refer to Green & Laws (1973) for a derivation as a constrained theory, although an extra constraint is needed in that derivation to obtain the present restricted theory.

With some of the notation of §2, the motion of the rod is now specified by

$$\boldsymbol{r} = \boldsymbol{r}(\zeta, t), \quad \boldsymbol{a}_3 = \boldsymbol{a}_3(\zeta, t) = \partial\boldsymbol{r}/\partial\zeta, \quad \boldsymbol{v} = \dot{\boldsymbol{r}}(\zeta, t) \quad (6.1)$$

and a rotation tensor

$$\boldsymbol{P} = \boldsymbol{a}_\alpha \otimes \boldsymbol{A}^\alpha + \boldsymbol{\nu} \otimes {}_{\mathrm{R}}\boldsymbol{\nu}, \quad \boldsymbol{a}_\alpha = \boldsymbol{P}\boldsymbol{A}_\alpha, \quad \boldsymbol{\nu} = \boldsymbol{P}\,{}_{\mathrm{R}}\boldsymbol{\nu} \quad (6.2)$$

with
$$\left.\begin{aligned} \boldsymbol{P}\boldsymbol{P}^{\mathrm{T}} = \boldsymbol{I}, \quad \dot{\boldsymbol{P}} = \boldsymbol{W}\boldsymbol{P}, \quad \boldsymbol{W}+\boldsymbol{W}^{\mathrm{T}} = \boldsymbol{0}, \\ \boldsymbol{W} = \dot{\boldsymbol{a}}_{\alpha} \otimes \boldsymbol{a}^{\alpha} + \dot{\boldsymbol{v}} \otimes \boldsymbol{v}. \end{aligned}\right\} \tag{6.3}$$

Let $\boldsymbol{\omega}$ be the axial vector corresponding to the skew tensor $\boldsymbol{W}$, so that
$$\boldsymbol{W}\boldsymbol{u} = \boldsymbol{\omega} \times \boldsymbol{u} \tag{6.4}$$
for all vectors $\boldsymbol{u}$. Let $\boldsymbol{\beta}$ be the axial vector corresponding to the skew tensor $\boldsymbol{P}^{\mathrm{T}}\,\partial\boldsymbol{P}/\partial\zeta$. Then
$$\boldsymbol{P}^{\mathrm{T}}\frac{\partial \boldsymbol{P}}{\partial \zeta}\boldsymbol{u} = \boldsymbol{\beta}\times\boldsymbol{u}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{P}^{\mathrm{T}}\frac{\partial \boldsymbol{\omega}}{\partial \zeta} \tag{6.5}$$
for all vectors $\boldsymbol{u}$.

Equations of mass conservation, momentum and moment of momentum are
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\zeta_1}^{\zeta_2} \rho\,\mathrm{d}s = 0, \quad \mathrm{d}s = a_{33}^{\frac{1}{2}}\,\mathrm{d}\zeta, \tag{6.6}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\zeta_1}^{\zeta_2} \rho\boldsymbol{v}\,\mathrm{d}s = \int_{\zeta_1}^{\zeta_2} \rho(\boldsymbol{f}+\boldsymbol{f}_{\mathrm{e}})\,\mathrm{d}s + [\boldsymbol{n}]_{\zeta_1}^{\zeta_2}, \tag{6.7}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\zeta_1}^{\zeta_2} \rho(\boldsymbol{r}\times\boldsymbol{v}+\boldsymbol{Y}\boldsymbol{\omega})\,\mathrm{d}s = \int_{\zeta_1}^{\zeta_2} \{\boldsymbol{r}\times(\boldsymbol{f}+\boldsymbol{f}_{\mathrm{e}})+\boldsymbol{l}+\boldsymbol{l}_{\mathrm{e}}+\boldsymbol{c}_{\mathrm{e}}\}\rho\,\mathrm{d}s + [\boldsymbol{r}\times\boldsymbol{n}+\boldsymbol{m}]_{\zeta_1}^{\zeta_2}, \tag{6.8}$$
where $\boldsymbol{f}$ is the assigned force, $\boldsymbol{l}$ is the assigned couple, $\boldsymbol{f}_{\mathrm{e}}$, $\boldsymbol{l}_{\mathrm{e}}$ are force and couple vectors due to the moments of the electromagnetic forces and $\boldsymbol{c}_{\mathrm{e}}$ is the electromagnetic couple. The inertia tensor $\boldsymbol{Y}$ is a function of ζ, independent of t. Field equations corresponding to (6.6)–(6.8) are
$$\left.\begin{aligned} \dot{\rho}+\rho\,\mathrm{div}_{\mathrm{e}}\,\boldsymbol{v} = 0 \quad \text{or} \quad \rho a_{33}^{\frac{1}{2}} = {}_{\mathrm{R}}\rho A_{33}^{\frac{1}{2}} = \lambda, \\ \lambda\dot{\boldsymbol{v}} = \lambda(\boldsymbol{f}+\boldsymbol{f}_{\mathrm{e}}) + \partial\boldsymbol{n}/\partial\zeta, \\ \lambda\boldsymbol{Y}\dot{\boldsymbol{\omega}} = \lambda(\boldsymbol{l}+\boldsymbol{l}_{\mathrm{e}})+\lambda\boldsymbol{c}_{\mathrm{e}}+\boldsymbol{a}_3\times\boldsymbol{n}+\partial\boldsymbol{m}/\partial\zeta. \end{aligned}\right\} \tag{6.9}$$

The equations of entropy balance still take the forms (2.15) but the energy equation is now
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\zeta_1}^{\zeta_2} (\epsilon+\tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}+\tfrac{1}{2}\boldsymbol{Y}\boldsymbol{\omega}\cdot\boldsymbol{\omega})\,\rho\,\mathrm{d}s &= \int_{\zeta_1}^{\zeta_2}\left\{r+\sum_{M+N=1}^{K} r_{MN}+(\boldsymbol{f}+\boldsymbol{f}_{\mathrm{e}})\cdot\boldsymbol{v}+(\boldsymbol{l}+\boldsymbol{l}_{\mathrm{e}})\cdot\boldsymbol{\omega}+\bar{w}\right\}\rho\,\mathrm{d}s \\ &\quad + \left[\boldsymbol{n}\cdot\boldsymbol{v}+\boldsymbol{m}\cdot\boldsymbol{\omega}-h-\sum_{M+N=1}^{P} h_{MN}\right]_{\zeta_1}^{\zeta_2}. \end{aligned} \tag{6.10}$$

The corresponding field equation is
$$\lambda(r+\sum r_{MN})-\lambda\dot{\epsilon}-\partial h/\partial\zeta-\sum\partial h_{MN}/\partial\zeta+\lambda\bar{w}-\lambda\boldsymbol{c}_{\mathrm{e}}\cdot\boldsymbol{\omega}+P = 0, \tag{6.11}$$

where
$$\left.\begin{aligned} P = \boldsymbol{n}\cdot\boldsymbol{a}_3\,\dot{c}+\boldsymbol{P}^{\mathrm{T}}\boldsymbol{m}\cdot\dot{\boldsymbol{\beta}}, \quad c = \tfrac{1}{2}\ln(a_{33}/A_{33}), \\ \lambda\bar{w}-\lambda\boldsymbol{c}_{\mathrm{e}}\cdot\boldsymbol{\omega} = P_{\mathrm{e}}+\sum_{M+N=0}^{L}(\hat{\boldsymbol{J}}_{MN}\cdot\tilde{\boldsymbol{E}}_{MN}+\tilde{\boldsymbol{E}}_{MN}\cdot\dot{\hat{\boldsymbol{D}}}_{MN}+\tilde{\boldsymbol{H}}_{MN}\cdot\dot{\hat{\boldsymbol{B}}}_{MN}), \\ P_{\mathrm{e}} = \boldsymbol{n}_{\mathrm{e}}\cdot\boldsymbol{a}_3\,\dot{c}+\boldsymbol{P}^{\mathrm{T}}\boldsymbol{m}_{\mathrm{e}}\cdot\dot{\boldsymbol{\beta}}. \end{aligned}\right\} \tag{6.12}$$

With the help of (2.17) and (4.1) we have an energy identity of the form (4.3) but with P, P_{e} now given by (6.12).

Discussion of a magnetic polarized thermoelastic rod now follows as in §4 except that the kinematic variables in (4.5) are replaced by

$$\boldsymbol{a}_3,\ \boldsymbol{P},\ \partial\boldsymbol{P}/\partial\zeta,\ \boldsymbol{A}_3, \tag{6.13}$$

or, taking into account the invariance of ψ under a constant rigid body rotation,

$$\psi = \psi_3(\theta, \theta_{MN}, c, \boldsymbol{\beta}, \boldsymbol{A}_3, \tilde{\boldsymbol{E}}_{MN}, \tilde{\boldsymbol{H}}_{MN}). \tag{6.14}$$

Expressions for the entropy densities and the electromagnetic vectors are the same form as those in (4.7), but (4.7 *f–h*) are replaced by

$$(\boldsymbol{n}+\boldsymbol{n}_{\mathrm{e}})\cdot\boldsymbol{a}_3 = \lambda\frac{\partial\psi_3}{\partial c}, \quad \boldsymbol{m}+\boldsymbol{m}_{\mathrm{e}} = \lambda\boldsymbol{P}\frac{\partial\psi_3}{\partial\boldsymbol{\beta}}. \tag{6.15}$$

To have component forms of constitutive equations corresponding to (4.11), we use the notation of (2.6) and note, from (6.2), that

$$\left.\begin{aligned} \boldsymbol{P}^{\mathrm{T}}\frac{\partial\boldsymbol{P}}{\partial\zeta} &= \kappa_{\beta\alpha}\boldsymbol{A}^{\alpha}\otimes\boldsymbol{A}^{\beta}+\kappa_{\alpha3}({}_{\mathrm{R}}\boldsymbol{\nu}\otimes\boldsymbol{A}^{\alpha}-\boldsymbol{A}^{\alpha}\otimes{}_{\mathrm{R}}\boldsymbol{\nu}),\\ \kappa_{\beta\alpha} = c_{\alpha\beta}-{}_{\mathrm{R}}c_{\alpha\beta} &= -\kappa_{\beta\alpha}, \quad \kappa_{\beta3} = c_{3\beta}a_{33}^{-\frac{1}{2}}-{}_{\mathrm{R}}c_{3\beta}A_{33}^{-\frac{1}{2}}, \end{aligned}\right\} \tag{6.16}$$

where the notations $\kappa_{\alpha\beta}$, $\kappa_{3\alpha}$ are different from the quantities defined in (4.8). Then, with (6.5), (6.15) and

$$\left.\begin{aligned} \boldsymbol{\beta} = \beta_{\alpha}\boldsymbol{A}^{\alpha}+\beta_3\,{}_{\mathrm{R}}\boldsymbol{\nu}, \quad \boldsymbol{n} = n_{\alpha}\boldsymbol{a}^{\alpha}+n_3\boldsymbol{\nu}, \quad \boldsymbol{n}_{\mathrm{e}} = n_{\mathrm{e}\alpha}\boldsymbol{a}^{\alpha}+n_{\mathrm{e}3}\boldsymbol{\nu},\\ \boldsymbol{m} = m_{\alpha}\boldsymbol{a}^{\alpha}+m_3\boldsymbol{\nu}, \quad \boldsymbol{m}_{\mathrm{e}} = m_{\mathrm{e}\alpha}\boldsymbol{a}^{\alpha}+m_{\mathrm{e}3}\boldsymbol{\nu} \end{aligned}\right\} \tag{6.17}$$

we have

$$\left.\begin{aligned} \beta_1 = \kappa_{23}, \quad \beta_2 = -\kappa_{13}, \quad \beta_3 = \kappa_{12} = -\kappa_{21},\\ n_3+n_{\mathrm{e}3} = \lambda\frac{\partial\psi_4}{\partial c}, \quad m_{\alpha}+m_{\mathrm{e}\alpha} = \lambda\frac{\partial\psi_4}{\partial\beta_{\alpha}}, \quad m_3+m_{\mathrm{e}3} = \lambda\frac{\partial\psi_4}{\partial\beta_3},\\ \psi = \psi_4(c, \beta_1, \beta_2, \beta_3, \theta, \theta_{MN}, \tilde{E}_{RSi}, \tilde{H}_{RSi}). \end{aligned}\right\} \tag{6.18}$$

The force components n_1, n_2 are not determined by constitutive equations.

In line with §5 we limit further discussion of the restricted theory to that of the linear theory of a rod which in its reference configuration is straight, with constant cross-sections. The reference line and its motion are specified by

$$\boldsymbol{R} = \zeta\boldsymbol{e}_3, \quad \boldsymbol{r} = \boldsymbol{R}+\boldsymbol{u}, \quad \boldsymbol{u} = u_i\boldsymbol{e}_i. \tag{6.19}$$

The linear rotation of each cross-section about $\boldsymbol{e}_3$ is $\bar{\delta}_{12} = -\bar{\delta}_{21}$ and

$$\left.\begin{aligned} c = \tfrac{1}{2}\gamma_{33} = \partial u_3/\partial\zeta, \quad \kappa_{12} = -\kappa_{21} = \partial\bar{\delta}_{12}/\partial\zeta,\\ \kappa_{\alpha3} = -\partial^2 u_{\alpha}/\partial\zeta^2, \quad \omega_1 = -\partial^2 u_2/\partial t\,\partial\zeta, \quad \omega_2 = \partial^2 u_1/\partial t\,\partial\zeta,\\ \omega_3 = \partial\bar{\delta}_{12}/\partial t, \quad \boldsymbol{\omega} = \omega_i\boldsymbol{e}_i. \end{aligned}\right\} \tag{6.20}$$

With quantities now referred to the reference body we have

$$\boldsymbol{n} = n_i\boldsymbol{e}_i, \quad \boldsymbol{m} = m_i\boldsymbol{e}_i, \quad \boldsymbol{f} = f_i\boldsymbol{e}_i, \text{ etc.} \tag{6.21}$$

and the linearized equations of motion are

$$\left.\begin{aligned} \rho \ddot{u}_i &= \rho f_i + \partial n_i/\partial \zeta, \\ \rho y^{22}\dot{\omega}_1 - \rho y^{12}\dot{\omega}_2 &= \rho l_1 - n_2 + \partial m_i/\partial \zeta, \\ \rho y^{11}\dot{\omega}_2 - \rho y^{12}\dot{\omega}_1 &= \rho l_2 + n_1 + \partial m_2/\partial \zeta, \\ \rho(y^{11}+y^{22})\,\dot{\omega}_3 &= \rho l_3 + \partial m_3/\partial \zeta \end{aligned}\right\} \tag{6.22}$$

where ρ is now reference density and $y^{\alpha\beta}$ are the same as the inertia coefficients in (5.5).

Since the rod in its reference configuration is at constant temperature, homogeneous, of constant density, and without electromagnetic fields, ψ is a quadratic function of the variables in (6.18), with constant coefficients. As in §5, to help with the specification of constitutive coefficients it is more convenient to express results in a partly inverted form. If

$$\left.\begin{aligned} \rho G &= \rho\psi - cn_3 - \beta_i m_i, \\ G &= G(n_3, m_i, \theta, \theta_\alpha, \tilde{E}_{MNi}, \tilde{H}_{MNi}), \end{aligned}\right\} \tag{6.23}$$

then, with $M+N = 0, 1$,

$$\left.\begin{aligned} &c = \tfrac{1}{2}\gamma_{33} = -\rho\frac{\partial G}{\partial n_3}, \quad \beta_i = -\rho\frac{\partial G}{\partial m_i}, \quad \eta = -\frac{\partial G}{\partial \theta}, \\ &\eta_\alpha = -\frac{\partial G}{\partial \theta_\alpha}, \quad \bar{D}_{MNi} = -\rho\frac{\partial G}{\partial \tilde{E}_{MNi}}, \quad B_{MNi} = -\rho\frac{\partial G}{\partial \tilde{H}_{MNi}} \end{aligned}\right\} \tag{6.24}$$

and G is a quadratic form. We limit attention to a restricted Cosserat rod which models a straight three-dimensional rod, with constant cross-sections, with the line of centroids along the ζ-axis, and with geometrical axes of symmetry in each section with respect to directions $\boldsymbol{e}_\alpha$. Also, in evaluating the coefficients in G we will again use the representation involving the functions (5.22), with the relations (5.23). Thus,

$$\begin{aligned} \rho G = {} & -\tfrac{1}{2}\bar{A}n_3^2 + \Sigma\,(C_k^{MN} n_3 \tilde{E}_{MNk} + F_k^{MN} n_3 \tilde{H}_{MNk}) - \bar{P}n_3\theta \\ & -\tfrac{1}{2}B_{ij}m_i m_j + \Sigma\,(C_{ik}^{MN} m_i \tilde{E}_{MNk} + F_{ik}^{MN} m_i \tilde{H}_{MNk}) \\ & -P_i^\alpha m_i\theta_\alpha - \tfrac{1}{2}P\theta^2 - \tfrac{1}{2}\bar{P}^{\alpha\beta}\theta_\alpha\theta_\beta \\ & +\Sigma\,(\tfrac{1}{2}L_{ij}^{MNRS}\tilde{E}_{MNi}\tilde{E}_{RSj} + M_{ij}^{MNRS}\tilde{E}_{MNi}\tilde{H}_{RSj} + \tfrac{1}{2}N_{ij}^{MNRS}\tilde{H}_{MNi}\tilde{H}_{RSj}) \\ & +\Sigma\,(R_i^{MN}\tilde{E}_{MNi}\theta + R_i^{MN\alpha}\tilde{E}_{MNi}\theta_\alpha + S_i^{MN}\tilde{H}_{MNi}\theta + S_i^{MN\alpha}\tilde{H}_{MNi}\theta_\alpha). \end{aligned} \tag{6.25}$$

The corresponding constitutive equations are

$$\left.\begin{aligned} c = \tfrac{1}{2}\gamma_{33} &= \bar{A}n_3 - \Sigma\,(C_k^{MN}\tilde{E}_{MNk} + F_k^{MN}\tilde{H}_{MNk}) + \bar{P}\theta, \\ \beta_i &= B_{ij}m_j - \Sigma\,(C_{ik}^{MN}\tilde{E}_{MNk} + F_{ik}^{MN}\tilde{H}_{MNk}) + P_i^\alpha\theta_\alpha, \\ \rho\eta &= \bar{P}n_3 + P\theta - \Sigma\,(R_i^{MN}\tilde{E}_{MNi} + S_i^{MN}\tilde{H}_{MNi}), \end{aligned}\right\} \tag{6.26}$$

$$\left.\begin{aligned}\rho\eta_\alpha &= P_i^\alpha m_i + \bar{P}^{\alpha\beta}\theta_\beta - \Sigma\,(R_i^{MN\alpha}\tilde{E}_{MNi} + S^{MN\alpha}\tilde{H}_{MNi}),\\ \bar{D}_{MNi} &= -C_i^{MN}n_3 - C_{ki}^{MN}m_k - \sum_{R,S}(L_{ij}^{MNRS}\tilde{E}_{RSj} + M_{ij}^{MNRS}\tilde{H}_{RSj}) - R_i^{MN}\theta - R_i^{MN\alpha}\theta_\alpha,\\ B_{MNi} &= -F_i^{MN}n_3 - F_{ki}^{MN}m_k - \sum_{R,S}(M_{ji}^{RSMN}\tilde{E}_{RSj} + N_{ij}^{MNRS}\tilde{H}_{RSj}) - S_i^{MN}\theta - S_i^{MN\alpha}\theta_\alpha.\end{aligned}\right\} \tag{6.26, cont.}$$

The constitutive coefficients are chosen to have the following values:

$$\left.\begin{gathered}\bar{A} = A^{-1}s_{3333},\quad \bar{P} = s_{33},\quad P = Ac^*,\\ \bar{P}^{11} = I_{11}c^*,\quad \bar{P}^{22} = I_{22}c^*,\quad \bar{P}^{12} = 0,\\ B_{11} = s_{3333}/I_{22},\quad B_{12} = 0,\quad B_{13} = -s_{1333}/I_{22},\\ B_{22} = s_{3333}/I_{11},\quad B_{23} = -s_{2333}/I_{11},\quad B_{33} = 1/\mathscr{D},\\ P_1^1 = 0,\quad P_2^1 = -s_{33},\quad P_3^1 = s_{23},\\ P_1^2 = s_{33},\quad P_2^2 = 0,\quad P_3^2 = -s_{13},\\ C_i^{00} = k_{33i}^*,\quad C_i^{10} = 0,\quad C_i^{01} = 0,\\ C_{ij}^{00} = 0,\quad C_{1j}^{10} = 0,\quad C_{2j}^{10} = -k_{33j}^*,\quad C_{3j}^{10} = k_{23j}^*,\\ C_{1j}^{01} = k_{33j}^*,\quad C_{2j}^{01} = 0,\quad C_{3j}^{01} = -k_{13j}^*,\\ F_i^{00} = l_{33i}^*,\quad F_i^{10} = 0,\quad F_i^{01} = 0,\\ F_{ij}^{00} = 0,\quad F_{1j}^{10} = 0,\quad F_{2j}^{10} = -l_{33j}^*,\quad F_{3j}^{10} = l_{23j}^*,\\ F_{1j}^{01} = l_{33j}^*,\quad F_{2j}^{01} = 0,\quad F_{3j}^{01} = -l_{13j}^*,\end{gathered}\right\} \tag{6.27}$$

the remaining coefficients having the same values as in (5.25).

7. Non-conducting rods

As one example of the theory of rods developed in previous sections, we consider a non-conducting rod in free space under isothermal conditions and in the absence of body force and applied tractions over the major surface of the rod. We use the restricted linear theory of §6, which models a straight rod with constant cross-sections, with line of centroids along the ζ-axis and geometrical axes of symmetry in each section with respect to two orthogonal directions $\boldsymbol{e}_\alpha$ defined earlier in §5. The effect of thermal variables is suppressed and the choice of electromagnetic variables is limited to E_{00i}, E_{10i}, E_{01i}, H_{00i}, H_{10i}, H_{01i}, where

$$\left.\begin{aligned}E_{00i} &= A\tilde{E}_{00i},\quad E_{10i} = I_{11}\tilde{E}_{10i},\quad E_{01i} = I_{22}\tilde{E}_{01i},\\ H_{00i} &= A\tilde{H}_{00i},\quad H_{10i} = I_{11}\tilde{H}_{10i},\quad H_{01i} = I_{22}\tilde{H}_{01i}.\end{aligned}\right\} \tag{7.1}$$

The linearized equations of motion (6.22) reduce to

$$\left.\begin{aligned}\rho\ddot{u}_i = \partial n_i/\partial\zeta,\quad -\rho y^{22}\,\partial^3 u_2/\partial t^2\,\partial\zeta = -n_2 + \partial m_1/\partial\zeta,\\ \rho y^{11}\,\partial^3 u_1/\partial t^2\,\partial\zeta = n_1 + \partial m_2/\partial\zeta,\quad \rho(y^{11}+y^{22})\,\partial^2\delta_{12}/\partial t^2 = \partial m_3/\partial\zeta\end{aligned}\right\} \tag{7.2}$$

and the quantities ρ, y^{11}, y^{22} are

$$\rho = A\rho^*, \quad \rho y^{11} = I_{11}\rho^*, \quad \rho y^{22} = I_{22}\rho^*. \tag{7.3}$$

The constitutive equations are given by (6.26) and (6.27) with the effect of thermal variables omitted. In addition, the relevant electromagnetic field equations are (5.10)–(5.13) with M, N taking the values $M, N = 0,0;\ 1,0;\ 0,1$.

When discussing deformation or vibration of the rod it is necessary to consider electromagnetic wave propagation in the free space surrounding the rod, and to use appropriate continuity conditions for the electromagnetic vectors at the surface of the rod. We do not embark on such a discussion here but limit our attention to a simpler situation in which the electromagnetic moduli for the rod are much greater than the moduli for free space so that we may adopt the approximate surface conditions

$$n_1\hat{B}^1 + n_2\hat{B}^2 = 0, \quad n_1\hat{D}^1 + n_2\hat{D}^2 = 0, \tag{7.4}$$

where n_1, n_2 are the components of the outward unit normal to the surface. From (7.4) and (B 24) it follows that

$$B'_{MN} = 0, \quad D'_{MN} = 0. \tag{7.5}$$

Let s be the arc length in the surface of the rod, normal to the axis, and E_s, H_s the tangential components along s of the electric and magnetic vectors. Then

$$E_s = n_1E_2 - n_2E_1, \quad H_s = n_1H_2 - n_2H_1 \tag{7.6}$$

and the surface conditions (7.4) imply that

$$\frac{\partial E_3}{\partial s} = \frac{\partial E_s}{\partial \zeta}, \quad \frac{\partial H_3}{\partial s} = \frac{\partial H_s}{\partial \zeta}. \tag{7.7}$$

With the help of (B 22) and (B 23), it follows from (7.7) that

$$\left.\begin{aligned} \frac{\partial E'_{003}}{\partial \zeta} &= 0, \quad \frac{\partial E'_{103}}{\partial \zeta} = E'_{001}, \quad \frac{\partial E'_{013}}{\partial \zeta} = E'_{002},\\ \frac{\partial H'_{003}}{\partial \zeta} &= 0, \quad \frac{\partial H'_{103}}{\partial \zeta} = H'_{001}, \quad \frac{\partial H'_{013}}{\partial \zeta} = H'_{002}. \end{aligned}\right\} \tag{7.8}$$

To make further progress, it is necessary to specify the shape of the cross-section of the rod. We consider a rod with elliptical cross-sections with semi-major and -minor axes (a, b). Also, in view of (B 17) and (B 18), we use the representations

$$\left.\begin{aligned} \hat{B}^i &= B_{00i}/A + \zeta^1 B_{10i}/I_{11} + \zeta^2 B_{01i}/I_{22},\\ \hat{D}^i &= \bar{D}_{00i}/A + \zeta^1 \bar{D}_{10i}/I_{11} + \zeta^2 \bar{D}_{01i}/I_{22},\\ A &= \pi ab, \quad I_{11} = \tfrac{1}{4}Aa^2, \quad I_{22} = \tfrac{1}{4}Ab^2. \end{aligned}\right\} \tag{7.9}$$

Then, applying the surface conditions (7.4) to the ellipse $(\zeta^1)^2/a^2 + (\zeta^2)^2/b^2 = 1$ gives

$$B_{001} = 0, \quad B_{002} = 0, \quad B_{101} = 0, \quad B_{012} = 0, \quad B_{011} + B_{102} = 0, \tag{7.10a–e}$$

$$\bar{D}_{001} = 0, \quad \bar{D}_{002} = 0, \quad \bar{D}_{101} = 0, \quad \bar{D}_{012} = 0, \quad \bar{D}_{011} + \bar{D}_{102} = 0. \tag{7.10f–j}$$

Also, since on the surface of the ellipse, $\zeta^1\,d\zeta^1/a^2+\zeta^2\,d\zeta^2/b^2=0$, it follows from (B 22) and (B 23) that

$$E'_{101}/a^2+E'_{012}/b^2=0, \quad H'_{101}/a^2+H'_{012}/b^2=0. \tag{7.11}$$

With the help of equations (7.8), (7.10), (7.11) and (5.10)–(5.13) for the specified values of M, N it is seen that the components of B_{MNi}, $\bar{D}_{MNi}$ – in addition to those in (7.10) – are all functions of t and independent of ζ, i.e.

$$B_{003}(t), \quad B_{103}(t), \quad B_{013}(t), \quad \bar{D}_{003}(t), \quad \bar{D}_{103}(t), \quad \bar{D}_{013}(t). \tag{7.12a–f}$$

Moreover, the only non-redundant electromagnetic equations (5.10)–(5.13) are

$$\frac{\partial E_{102}}{\partial \zeta}+E'_{101}=0, \quad \frac{\partial H_{102}}{\partial \zeta}+H'_{101}=0, \tag{7.13a, b}$$

$$\frac{\partial E_{012}}{\partial \zeta}+E'_{011}-\frac{\partial E_{101}}{\partial \zeta}+E'_{102}=0, \tag{7.13c}$$

$$\frac{\partial H_{012}}{\partial \zeta}+H'_{011}-\frac{\partial H_{101}}{\partial \zeta}+H'_{102}=0, \tag{7.13d}$$

$$\dot{B}_{102}=-\frac{\partial E_{101}}{\partial \zeta}+E'_{102}-E_{003}, \tag{7.13e}$$

$$-\dot{\bar{D}}_{102}=-\frac{\partial H_{101}}{\partial \zeta}+H'_{102}-H_{003}, \tag{7.13f}$$

$$\dot{B}_{013}=E'_{013}-E_{001}, \quad -\dot{\bar{D}}_{013}=H'_{013}-H_{001}, \tag{7.13g, h}$$

$$\dot{B}_{103}=E'_{103}+E_{002}, \quad -\dot{\bar{D}}_{103}=H'_{103}+H_{002}, \tag{7.13i, j}$$

$$\frac{\partial}{\partial \zeta}\left(\frac{E_{102}}{a^2}-\frac{E_{011}}{b^2}\right)=0, \quad \frac{\partial}{\partial \zeta}\left(\frac{H_{102}}{a^2}-\frac{H_{011}}{b^2}\right)=0. \tag{7.13h, l}$$

The relevant constitutive equations are given by (6.26), (6.27) and part of (5.25), with thermal variables omitted.

The problem has now been reduced to equations (7.2), (7.10), (7.12) and (7.13k, l) for the variables u_i, $\bar{\delta}_{12}$, E_{00i}, E_{10i}, E_{01i}, H_{00i}, H_{10i}, H_{01i} and then equations (7.8), (7.11), (7.13a–j) for E'_{00i}, E'_{10i}, E'_{01i}, H'_{00i}, H'_{10i}, H'_{01i}.

As an example, suppose that the rod is maintained in equilibrium by constant couples applied over the ends of the rod. Then, with

$$u_3=0, \quad n_i=0, \quad m_i=G_i, \tag{7.14}$$

where G_i are constants, equations (7.2) are satisfied. All the quantities in (7.12) are zero and from (7.10a, b, f, g), (7.12a, d) and the constitutive equations (6.26) and (6.27), we obtain

$$E_{00i}=0, \quad H_{00i}=0, \quad c=0. \tag{7.15}$$

Again, from (7.10), (7.12) and the constitutive equations we have

$$L_{ij}^{1010}\,\tilde{E}_{10j}+M_{ij}^{1010}\,\tilde{H}_{10j}+C_{2i}^{10}\,m_2+C_{3i}^{10}\,m_3=0, \tag{7.16a}$$

$$L_{ij}^{0101}\,\tilde{E}_{01j}+M_{ij}^{0101}\,\tilde{H}_{01j}+C_{1i}^{01}\,m_1+C_{3i}^{01}\,m_3=0, \tag{7.16b}$$

$$M_{ji}^{1010}\,\tilde{E}_{10j}+N_{ij}^{1010}\,\tilde{H}_{10j}+F_{2i}^{10}\,m_2+F_{3i}^{10}\,m_3=0, \tag{7.16c}$$

$$M_{ji}^{0101}\,\tilde{E}_{01j}+N_{ji}^{0101}\,\tilde{H}_{01j}+F_{1i}^{01}\,m_1+F_{3i}^{01}\,m_3=0, \tag{7.16d}$$

for $i = 1, 3$ in (7.16a, c) and $i = 2, 3$ in (7.16b, d) and

$$\left.\begin{aligned} L^{1010}_{2j}\tilde{E}_{10j}+L^{0101}_{1j}\tilde{E}_{01j}+M^{1010}_{2j}\tilde{H}_{10j}+M^{0101}_{1j}\tilde{H}_{01j}+C^{10}_{22}m_2+C^{01}_{11}m_1+(C^{10}_{32}+C^{01}_{31})\,m_3 = 0,\\ M^{1010}_{j2}\tilde{E}_{10j}+M^{0101}_{j1}\tilde{E}_{01j}+N^{1010}_{2j}\tilde{H}_{10j}+N^{0101}_{1j}\tilde{H}_{01j}+F^{10}_{22}m_2+F^{01}_{11}m_1+(F^{10}_{32}+F^{01}_{31})\,m_3 = 0.\end{aligned}\right\} \quad (7.17)$$

Also, from (7.13k, l) we obtain

$$b^2E_{102} = a^2E_{011}, \quad b^2H_{102} = a^2H_{011}. \quad (7.18)$$

The twelve equations (7.16), (7.17) and (7.18) may be used to express $\tilde{E}_{10i}$, $\tilde{E}_{01i}$, $\tilde{H}_{10i}$, $\tilde{H}_{01i}$ in terms of m_1, m_2, m_3. Then, from (6.26) the expressions for curvatures and torsion of the rod are given by

$$\left.\begin{aligned} \beta_1 &= B_{11}m_1+B_{13}m_3-C^{01}_{1i}\tilde{E}_{01i}-F^{01}_{1i}\tilde{H}_{01i},\\ \beta_2 &= B_{22}m_2+B_{23}m_3-C^{10}_{2i}\tilde{E}_{10i}-F^{10}_{2i}\tilde{H}_{10i},\\ \beta_3 &= B_{13}m_1+B_{23}m_2+B_{33}m_3-C^{10}_{3i}\tilde{E}_{10i}-C^{01}_{3i}\tilde{E}_{01i}-F^{10}_{3i}\tilde{H}_{10i}-F^{01}_{3i}\tilde{H}_{01i},\\ \beta_1 &= -\partial^2u_2/\partial\zeta^2, \quad \beta_2 = \partial^2u_1/\partial\zeta^2, \quad \beta_3 = \partial\bar{\delta}_{12}/\partial\zeta.\end{aligned}\right\} \quad (7.19)$$

As a second example, consider wave propagation along the rod. If we assume that all variables are proportional to $\exp\mathrm{i}(mz+\omega t)$ and remove the exponential factor, then equations (7.2) yield

$$\left.\begin{aligned} -\rho\omega^2u_3 = \mathrm{i}mn_3, \quad \rho\omega^2(1+y^{11}m^2)\,u_1 = -m^2m_2,\\ \rho\omega^2(1+y^{22}m^2)\,u_2 = m^2m_1, \quad \rho\omega^2(y^{11}+y^{22})\,m^2\bar{\delta}_{12} = -\mathrm{i}mm_3.\end{aligned}\right\} \quad (7.20)$$

The functions in (7.12) are again zero so that equations (7.16), (7.17) and (7.18) are satisfied, and from (6.26) we have

$$\left.\begin{aligned} L^{0000}_{ij}\tilde{E}_{00j}+M^{0000}_{ij}\tilde{H}_{00j}+C^{00}_{i}n_3 &= 0,\\ M^{0000}_{ji}\tilde{E}_{00j}+N^{0000}_{ij}\tilde{H}_{00j}+F^{00}_{i}n_3 &= 0,\\ c = \mathrm{i}mu_3 &= \tilde{A}n_3-C^{00}_{i}\tilde{E}_{00i}-F^{00}_{i}\tilde{H}_{00i}.\end{aligned}\right\} \quad (7.21)$$

In addition, equations (7.19) hold with

$$\beta_1 = m^2u_2, \quad \beta_2 = -m^2u_1, \quad \beta_3 = \mathrm{i}m\bar{\delta}_{12}.$$

The frequencies of propagation may be found in the usual way, yielding one extensional frequency and three frequencies involving flexure and torsion. The surface values of the electromagnetic field may then be found from equations (7.8), (7.11) and (7.13).

The problem of a non-conducting rod in free space may also be discussed with the aid of the theory of §5. The electromagnetic part of the theory is the same as that used in this §7 and reduces to equations (7.8), (7.10), (7.11), (7.12) and (7.13). Mechanical equations of motion and the constitutive equations are then obtained from §5.

8. Piezoelectric crystal rods

Isothermal forced vibrations of piezoelectric crystal rods may be studied as a special case of the theory of §§5 or 6. We again limit our attention to the theory of §6 which models a straight rod, with constant cross-sections, with line of centroids along the ζ-axis, and with geometrical axes of symmetry in each section with respect to orthogonal directions $\boldsymbol{e}_\alpha$. The linear equations of motion (6.22) again reduce to (7.2) except that externally applied tractions may also be included if desired. In piezoelectric theory, the magnetic fields H_{MNi} or $\tilde{H}_{MNi}$ are absent from all constitutive equations so that, from (6.26), $B_{MNi} = 0$. With thermal variables omitted and the electromagnetic fields limited to

$$E_{00i} = A\tilde{E}_{00i}, \quad E_{10i} = I_{11}\tilde{E}_{10i}, \quad E_{01i} = I_{22}\tilde{E}_{01i}, \tag{8.1}$$

the constitutive equations (6.26) and (6.27) reduce to

$$\left.\begin{aligned} c &= \bar{A}n_3 - C_i^{00}\tilde{E}_{00i},\\ \beta_1 &= B_{11}m_1 + B_{13}m_3 - C_{1i}^{01}\tilde{E}_{01i},\\ \beta_2 &= B_{22}m_2 + B_{23}m_3 - C_{2i}^{10}\tilde{E}_{10i},\\ \beta_3 &= B_{13}m_1 + B_{23}m_2 + B_{33}m_3 - C_{3i}^{10}\tilde{E}_{10i} - C_{3i}^{01}\tilde{E}_{01i},\\ \bar{D}_{00i} &= -C_i^{00}n_3 - L_{ij}^{0000}\tilde{E}_{00j},\\ \bar{D}_{10i} &= -C_{2i}^{10}m_2 - C_{3i}^{10}m_3 - L_{ij}^{1010}\tilde{E}_{10j},\\ \bar{D}_{01i} &= -C_{1i}^{01}m_1 - C_{3i}^{01}m_3 - L_{ij}^{0101}\tilde{E}_{01j}. \end{aligned}\right\} \tag{8.2}$$

From (5.10)–(5.13) and (5.22) we see that the appropriate electromagnetic field equations are

$$-D'_{00} = \frac{\partial \bar{D}_{003}}{\partial \zeta}, \quad -D'_{10} + \bar{D}_{001} = \frac{\partial \bar{D}_{103}}{\partial \zeta}, \quad -D'_{01} + \bar{D}_{002} = \frac{\partial \bar{D}_{013}}{\partial \zeta}, \tag{8.3 a–c}$$

$$\frac{\partial E_{002}}{\partial \zeta} + E'_{001} = 0, \quad -\frac{\partial E_{001}}{\partial \zeta} + E'_{002} = 0, \quad E'_{003} = 0, \tag{8.3 d–f}$$

$$\frac{\partial E_{102}}{\partial \zeta} + E'_{101} = 0, \quad -\frac{\partial E_{101}}{\partial \zeta} + E'_{102} - E_{003} = 0, \quad E'_{103} + E_{002} = 0, \tag{8.3 g–i}$$

$$\frac{\partial E_{012}}{\partial \zeta} + E'_{011} + E_{003} = 0, \quad -\frac{\partial E_{011}}{\partial \zeta} + E'_{012} = 0, \quad E'_{013} - E_{001} = 0. \tag{8.3 j–l}$$

Let ϕ be the applied potential at the major surface of the rod. Then, from (8.3 d–l) we have

$$\left.\begin{aligned} &E_{001} = -\Phi_{01}, \quad E_{002} = \Phi_{10}, \quad E_{102} = \Phi_{11},\\ &E_{101} = E_{012} - \Phi_{12} - \Phi_{21}, \quad E_{011} = -\Phi_{22}, \quad E_{003} = -\partial(E_{012} - \Phi_{21})/\partial\zeta, \end{aligned}\right\} \tag{8.4}$$

where

$$\left.\begin{aligned} \Phi_{10} &= \oint \phi \, d\zeta^1, & \Phi_{01} &= \oint \phi \, d\zeta^2,\\ \Phi_{11} &= \oint \phi\zeta^1 \, d\zeta^1, & \Phi_{12} &= \oint \phi\zeta^1 \, d\zeta^2,\\ \Phi_{21} &= \oint \phi\zeta^2 \, d\zeta^1, & \Phi_{22} &= \oint \phi\zeta^2 \, d\zeta^2. \end{aligned}\right\} \tag{8.5}$$

In view of (8.4) we are left with three dependent variables E_{103}, E_{013}, E_{012}. To complete the differential equations (8.3 a–c) for these variables we make use of the representation

$$\bar{D}^i = \bar{D}_{00i}/A + \zeta^1\bar{D}_{10i}/I_{11} + \zeta^2\bar{D}_{01i}/I_{22} \tag{8.6}$$

so that, with the help of (B 24), we have

$$D'_{00} = A(\bar{D}_{101}/I_{11}+\bar{D}_{012}/I_{22}), \quad D'_{10} = \bar{D}_{001}, \quad D'_{01} = \bar{D}_{002}. \tag{8.7}$$

Then, with (8.7), the differential equations (8.3 *a–e*) reduce to

$$\frac{\partial \bar{D}_{003}}{\partial \zeta} = -A\left(\frac{\bar{D}_{101}}{I_{11}}+\frac{\bar{D}_{012}}{I_{22}}\right), \quad \bar{D}_{103} = \bar{D}_{103}(t), \quad \bar{D}_{013} = \bar{D}_{013}(t). \tag{8.8}$$

9. Alternative theory for rods with rectangular cross-section

So far we have interpreted both the general theory of § 3 and the restricted theory of § 6 with the help of polynomial representations and formulae in Appendix B. For some types of electromagnetic surface conditions and for some cross-sections of the rod, it is more convenient to interpret the one-dimensional theory with the help of functions other than polynomials in ζ^1, ζ^2 as far as the electromagnetic part of the theory is concerned. To elaborate, we consider a uniform rod of constant rectangular section whose sides are of lengths a, b, the surfaces of the rod being defined by

$$\zeta^1 = \pm\tfrac{1}{2}a, \quad -\tfrac{1}{2}b \leqslant \zeta^2 \leqslant \tfrac{1}{2}b \quad \text{and} \quad \zeta^2 = \pm\tfrac{1}{2}b, \quad -\tfrac{1}{2}a \leqslant \zeta^1 \leqslant \tfrac{1}{2}a. \tag{9.1}$$

Then, in Appendix B, we choose

$$\left.\begin{aligned}
\psi^M(\zeta^1) &= (2/a)^{\frac{1}{2}} \sin\tfrac{1}{2}M\pi(1+2\zeta^1/a),\\
\chi^M(\zeta^1) &= (2/a)^{\frac{1}{2}} \cos\tfrac{1}{2}M\pi(1+2\zeta^1/a) \quad (M\neq 0), \quad \chi^0(\zeta^1) = a^{-\frac{1}{2}},\\
\bar{\psi}^N(\zeta^2) &= (2/b)^{\frac{1}{2}} \sin\tfrac{1}{2}N\pi(1+2\zeta^2/b),\\
\bar{\chi}^N(\zeta^2) &= (2/b)^{\frac{1}{2}} \cos\tfrac{1}{2}N\pi(1+2\zeta^2/b) \quad (N\neq 0), \quad \bar{\chi}^0(\zeta^2) = b^{-\frac{1}{2}},\\
\psi^M_M = M\pi/a, \quad \bar{\psi}^N_N &= N\pi/b, \quad \psi^M_K = 0, \quad \bar{\psi}^N_K = 0 \quad (M\neq K, N\neq K),\\
\chi^M_M = -M\pi/a, \quad \bar{\chi}^N_N &= -N\pi/b, \quad \chi^M_K = 0, \quad \bar{\chi}^N_K = 0 \quad (M\neq K, N\neq K).
\end{aligned}\right\} \tag{9.2}$$

In view of (B 26), (B 27) and (B 28), it follows from (9.2) that

$$E_{MNi} = \tilde{E}_{MNi}, \quad H_{MNi} = \tilde{H}_{MNi}, \tag{9.3 a, b}$$

$$\tilde{E}_{M01} = 0, \quad \tilde{E}_{M03} = 0, \quad \tilde{H}_{M02} = 0 \quad (M = 0, 1, \ldots), \tag{9.3 c–e}$$

$$\tilde{E}_{0N2} = 0, \quad \tilde{E}_{0N3} = 0, \quad \tilde{H}_{0N1} = 0 \quad (N = 0, 1, \ldots). \tag{9.3 f–h}$$

With representations (9.2), some changes are necessary in the values of the constitutive coefficients in (5.25), (5.26) and (6.27). Here we record only the changes in values of the constitutive coefficients for the restricted theory in (6.27). The coefficients $\bar{A}$, B_{ij}, $\bar{P}$, P, $\bar{P}^{\alpha\beta}$, P^{α}_i are unaltered, while the remaining coefficients now have the values

$$\left.\begin{aligned}
&C^{0N}_1 = \frac{2^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}N\pi}\{1-(-1)^N\}k^*_{331} \quad (N\neq 0), \quad C^{MN}_i = 0 \quad (M\neq 0), \quad C^{00}_1 = 0,\\
&C^{M0}_2 = \frac{2^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}M\pi}\{1-(-1)^M\}k^*_{332} \quad (M\neq 0), \quad C^{MN}_2 = 0 \quad (N\neq 0), \quad C^{00}_2 = 0,\\
&C^{MN}_3 = \frac{2}{(ab)^{\frac{1}{2}}MN\pi^2}\{1-(-1)^M\}\{1-(-1)^N\}k^*_{333} \quad (M\neq 0,\ N\neq 0),\\
&C^{M0}_3 = 0, \quad C^{0N}_3 = 0,
\end{aligned}\right\} \tag{9.4 a}$$

$$\left.\begin{aligned}
&F_1^{M0} = \frac{2^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}M\pi}\{1-(-1)^M\}\,l^*_{331} \quad (M \neq 0), \quad F_1^{MN} = 0 \quad (N \neq 0), \quad F_1^{00} = 0,\\
&F_2^{0N} = \frac{2^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}N\pi}\{1-(-1)^N\}\,l^*_{332} \quad (N \neq 0), \quad F_2^{MN} = 0 \quad (M \neq 0), \quad F_2^{00} = 0,\\
&F_3^{00} = \frac{l^*_{333}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}, \quad F_3^{MN} = 0 \quad (M \neq 0,\ N \neq 0),
\end{aligned}\right\} \tag{9.4b}$$

$$\left.\begin{aligned}
&C_{11}^{0N} = -\frac{a^{\frac{1}{2}}b^{\frac{3}{2}}}{2^{\frac{1}{2}}N\pi}\{1+(-1)^N\}\frac{k^*_{331}}{I_{22}} \quad (N \neq 0), \quad C_{11}^{MN} = 0 \quad (M \neq 0), \quad C_{11}^{00} = 0,\\
&C_{21}^{MN} = \frac{2a^{\frac{3}{2}}b^{\frac{1}{2}}}{M^2N\pi^3}\{1-(-1)^M\}\{1-(-1)^N\}\frac{k^*_{331}}{I_{11}} \quad (M \neq 0,\ N \neq 0),\\
&C_{21}^{0N} = 0, \quad C_{21}^{M0} = 0,\\
&C_{12}^{MN} = -\frac{2a^{\frac{1}{2}}b^{\frac{3}{2}}}{MN^2\pi^3}\{1-(-1)^M\}\{1-(-1)^N\}\frac{k^*_{332}}{I_{22}} \quad (M \neq 0,\ N \neq 0),\\
&C_{12}^{M0} = 0, \quad C_{12}^{0N} = 0,\\
&C_{22}^{M0} = \frac{a^{\frac{3}{2}}b^{\frac{1}{2}}}{2^{\frac{1}{2}}M\pi}\{1+(-1)^M\}\frac{k^*_{332}}{I_{11}} \quad (M \neq 0), \quad C_{22}^{MN} = 0 \quad (N \neq 0), \quad C_{22}^{00} = 0,
\end{aligned}\right\} \tag{9.4c}$$

$$\left.\begin{aligned}
&C_{13}^{MN} = -\frac{a^{\frac{1}{2}}b^{\frac{3}{2}}}{MN\pi^2}\{1-(-1)^M\}\{1+(-1)^N\}\frac{k^*_{333}}{I_{22}} \quad (M \neq 0,\ N \neq 0), \quad C_{13}^{M0} = 0, \quad C_{13}^{0N} = 0,\\
&C_{23}^{MN} = \frac{a^{\frac{3}{2}}b^{\frac{1}{2}}}{MN\pi^2}\{1+(-1)^M\}\{1-(-1)^N\}\frac{k^*_{333}}{I_{11}} \quad (M \neq 0,\ N \neq 0), \quad C_{23}^{M0} = 0, \quad C_{23}^{0N} = 0,\\
&C_{31}^{MN} = -\frac{2a^{\frac{3}{2}}b^{\frac{1}{2}}}{M^2N\pi^3}\{1-(-1)^M\}\{1-(-1)^N\}\frac{k^*_{231}}{I_{11}} \quad (M \neq 0,\ N \neq 0),\\
&C_{31}^{0N} = \frac{a^{\frac{1}{2}}b^{\frac{3}{2}}}{2^{\frac{1}{2}}N\pi}\{1+(-1)^N\}\frac{k^*_{131}}{I_{22}} \quad (N \neq 0), \quad C_{31}^{00} = 0, \quad C_{31}^{M0} = 0,\\
&C_{32}^{MN} = \frac{2a^{\frac{1}{2}}b^{\frac{3}{2}}}{MN^2\pi^3}\{1-(-1)^M\}\{1-(-1)^N\}\frac{k^*_{132}}{I_{22}} \quad (M \neq 0,\ N \neq 0),\\
&C_{32}^{M0} = -\frac{a^{\frac{3}{2}}b^{\frac{1}{2}}}{2^{\frac{1}{2}}M\pi}\{1+(-1)^M\}\frac{k^*_{232}}{I_{11}} \quad (M \neq 0), \quad C_{32}^{00} = 0, \quad C_{32}^{0N} = 0,
\end{aligned}\right\} \tag{9.4d}$$

$$\left.\begin{aligned}
C_{33}^{MN} = &-\frac{a^{\frac{3}{2}}b^{\frac{1}{2}}}{MN\pi^2}\{1+(-1)^M\}\{1-(-1)^N\}\frac{k^*_{233}}{I_{11}}\\
&+\frac{a^{\frac{1}{2}}b^{\frac{3}{2}}}{MN\pi^2}\{1-(-1)^M\}\{1+(-1)^N\}\frac{k^*_{133}}{I_{22}} \quad (M \neq 0,\ N \neq 0),\\
C_{33}^{M0} = 0,& \quad C_{33}^{0N} = 0,
\end{aligned}\right\} \tag{9.4e}$$

$$\left.\begin{aligned}
&F_{11}^{MN} = -\frac{2a^{\frac{1}{2}}b^{\frac{3}{2}}}{MN^2\pi^3}\{1-(-1)^M\}\{1-(-1)^N\}\frac{l^*_{331}}{I_{22}} \quad (M \neq 0,\ N \neq 0),\\
&F_{11}^{0N} = 0, \quad F_{11}^{M0} = 0,
\end{aligned}\right\} \tag{9.4f}$$

$$\left.\begin{aligned}
&F_{21}^{M0} = \frac{a^{\frac{3}{2}}b^{\frac{1}{2}}}{2^{\frac{1}{2}}M\pi}\{1+(-1)^M\}\frac{l^*_{331}}{I_{11}} \quad (M\neq 0), \quad F_{21}^{MN}=0 \quad (N\neq 0),\\
&F_{12}^{0N} = -\frac{a^{\frac{1}{2}}b^{\frac{3}{2}}}{2^{\frac{1}{2}}N\pi}\{1+(-1)^N\}\frac{l^*_{332}}{I_{22}} \quad (N\neq 0), \quad F_{12}^{MN}=0 \quad (M\neq 0), \quad F_{12}^{00}=0,\\
&F_{22}^{MN} = \frac{2a^{\frac{3}{2}}b^{\frac{1}{2}}}{M^2N\pi^3}\{1-(-1)^M\}\{1-(-1)^N\}\frac{l^*_{332}}{I_{11}} \quad (M\neq 0,\ N\neq 0),\\
&F_{22}^{0N}=0, \quad F_{22}^{M0}=0,
\end{aligned}\right\} \qquad (9.4f,\ cont.)$$

$$\left.\begin{aligned}
&F_{13}^{0N} = -\frac{2^{\frac{1}{2}}a^{\frac{1}{2}}b^{\frac{3}{2}}}{N^2\pi^2}\{1-(-1)^N\}\frac{l^*_{333}}{I_{22}} \quad (N\neq 0), \quad F_{13}^{MN}=0 \quad (M\neq 0), \quad F_{13}^{00}=0,\\
&F_{23}^{M0} = \frac{2^{\frac{1}{2}}a^{\frac{3}{2}}b^{\frac{1}{2}}}{M^2\pi^2}\{1-(-1)^M\}\frac{l^*_{333}}{I_{11}} \quad (M\neq 0), \quad F_{23}^{MN}=0 \quad (N\neq 0), \quad F_{23}^{00}=0,\\
&F_{31}^{M0} = -\frac{a^{\frac{3}{2}}b^{\frac{1}{2}}}{2^{\frac{1}{2}}M\pi}\{1+(-1)^M\}\frac{l^*_{231}}{I_{11}} \quad (M\neq 0), \quad F_{31}^{0N}=0, \quad F_{31}^{00}=0,\\
&F_{31}^{MN} = \frac{2a^{\frac{1}{2}}b^{\frac{3}{2}}}{MN^2\pi^3}\{1-(-1)^M\}\{1-(-1)^N\}\frac{l^*_{131}}{I_{22}} \quad (M\neq 0,\ N\neq 0),\\
&F_{32}^{MN} = -\frac{2a^{\frac{3}{2}}b^{\frac{1}{2}}}{M^2N\pi^3}\{1-(-1)^M\}\{1-(-1)^N\}\frac{l^*_{232}}{I_{11}} \quad (M\neq 0,\ N\neq 0),
\end{aligned}\right\} \qquad (9.4g)$$

$$\left.\begin{aligned}
&F_{32}^{0N} = \frac{a^{\frac{1}{2}}b^{\frac{3}{2}}}{2^{\frac{1}{2}}N\pi}\{1+(-1)^N\}\frac{l^*_{132}}{I_{22}} \quad (N\neq 0), \quad F_{32}^{M0}=0,\\
&F_{33}^{M0} = -\frac{2^{\frac{1}{2}}a^{\frac{3}{2}}b^{\frac{1}{2}}}{M^2\pi^2}\{1-(-1)^M\}\frac{l^*_{233}}{I_{11}} \quad (M\neq 0),\\
&F_{33}^{0N} = \frac{2^{\frac{1}{2}}a^{\frac{1}{2}}b^{\frac{3}{2}}}{N^2\pi^2}\{1-(-1)^N\}\frac{l^*_{133}}{I_{22}} \quad (N\neq 0),\\
&F_{33}^{MN}=0 \quad (M\neq 0,\ N\neq 0), \quad F_{33}^{00}=0,\\
&R_i^{MN}k^*_{33i} = Af^*_i C_i^{MN}, \quad R_i^{MN1}k^*_{33i} = -I_{11}C_{2i}^{MN}f^*_i,\\
&R_i^{MN2}k^*_{33i} = I_{22}C_{1i}^{MN}f^*_i, \quad S_i^{MN}l^*_{33i} = Ag^*_i F_i^{MN},\\
&S_i^{MN1}l^*_{33i} = -I_{11}F_{2i}^{MN}g^*_i, \quad S_i^{MN2}l^*_{33i} = I_{22}F_{1i}^{MN}g^*_i,
\end{aligned}\right\} \qquad (9.4h)$$

where in the last six formulae there is no summation over the repeated suffix i, and

$$\left.\begin{aligned}
&L_{11}^{MNRS} = \delta^{MR}\delta^{NS}f^*_{11} \quad (N\neq 0,\ S\neq 0), \quad L_{11}^{M0RS}=0, \quad L_{11}^{MNR0}=0,\\
&L_{12}^{MNRS} = \frac{4RN\{1-(-1)^{M+R}\}\{1-(-1)^{N+S}\}f^*_{12}}{(R^2-M^2)(N^2-S^2)\pi^2} \quad (R\neq M,\ S\neq N,\ M,N,R,S\neq 0),\\
&L_{12}^{MNR0} = \frac{2^{\frac{3}{2}}R\{1-(-1)^{M+R}\}\{1-(-1)^N\}f^*_{12}}{(R^2-M^2)N\pi^2} \quad (R\neq M,\ M,R,N\neq 0),\\
&L_{12}^{0NRS} = \frac{2^{\frac{3}{2}}N\{1-(-1)^R\}\{1-(-1)^{N+S}\}f^*_{12}}{R(N^2-S^2)\pi^2} \quad (S\neq N,\ R,N,S\neq 0),
\end{aligned}\right\} \qquad (9.4i)$$

$$\left.\begin{aligned}
&L_{12}^{0NR0} = \frac{2\{1-(-1)^R\}\{1-(-1)^N\}f_{12}^*}{RN\pi^2} \quad (R, N \neq 0),\\
&L_{12}^{M0RS} = 0, \quad L_{12}^{MN0S} = 0, \quad L_{12}^{MNRS} = 0 \quad (M, N, S \neq 0),\\
&L_{12}^{MNRN} = 0 \quad (M, N, R \neq 0), \quad L_{12}^{MNM0} = 0 \quad (M, N \neq 0), \quad L_{12}^{0NRN} = 0 \quad (R, N \neq 0),\\
&L_{13}^{MNRS} = \frac{2R\{1-(-1)^{M+R}\}\delta^{NS}f_{13}^*}{(R^2-M^2)\pi} \quad (R \neq M,\ R, M, N, S \neq 0),\\
&L_{13}^{0NRS} = \frac{2^{\frac{1}{2}}\{1-(-1)^R\}\delta^{NS}f_{13}^*}{R\pi} \quad (R, N, S \neq 0),\\
&L_{13}^{M0RS} = 0, \quad L_{13}^{MN0S} = 0, \quad L_{13}^{MNR0} = 0; \quad L_{13}^{MNMS} = 0 \quad (M, N, S \neq 0),\\
&L_{22}^{MNRS} = \delta^{MR}\delta^{NS}f_{22}^* \quad (M \neq 0,\ R \neq 0), \quad L_{22}^{0NRS} = 0, \quad L_{22}^{MN0S} = 0,\\
&L_{23}^{MNRS} = \frac{2S\{1-(-1)^{N+S}\}\delta^{RM}f_{23}^*}{(S^2-N^2)\pi} \quad (N \neq S,\ M, R, N, S \neq 0),\\
&L_{23}^{M0RS} = \frac{2^{\frac{1}{2}}\{1-(-1)^S\}\delta^{RM}f_{23}^*}{S\pi} \quad (M, R, S \neq 0),\\
&L_{23}^{MNRN} = 0 \quad (M, R, N \neq 0), \quad L_{23}^{0NRS} = 0, \quad L_{23}^{MN0S} = 0, \quad L_{23}^{MNR0} = 0,\\
&L_{33}^{MNRS} = \delta^{RM}\delta^{SN}f_{33}^* \quad (M, R, N, S \neq 0), \quad L_{33}^{0NRS} = 0, \quad L_{33}^{M0RS} = 0, \quad L_{33}^{MN0S} = 0,\\
&\qquad L_{33}^{MNR0} = 0,
\end{aligned}\right\} \quad (9.4i,\ cont.)$$

$$\left.\begin{aligned}
&N_{11}^{MNRS} = \delta^{MR}\delta^{NS}g_{11}^* \quad (M, R \neq 0), \quad N_{11}^{0NRS} = 0, \quad N_{11}^{MN0S} = 0,\\
&N_{12}^{MNRS} = \frac{4MS\{1-(-1)^{M+R}\}\{1-(-1)^{N+S}\}g_{12}^*}{(M^2-R^2)(S^2-N^2)\pi^2} \quad (M \neq R,\ N \neq S,\ M, R, N, S \neq 0),\\
&N_{12}^{M0RS} = \frac{2^{\frac{3}{2}}M\{1-(-1)^{M+R}\}\{1-(-1)^S\}g_{12}^*}{(M^2-R^2)S\pi^2} \quad (M \neq R,\ M, R, S \neq 0),\\
&N_{12}^{M00S} = \frac{2\{1-(-1)^M\}\{1-(-1)^S\}g_{12}^*}{MS\pi^2} \quad (M, S \neq 0),\\
&N_{12}^{MN0S} = \frac{2^{\frac{3}{2}}S\{1-(-1)^M\}\{1-(-1)^{N+S}\}g_{12}^*}{M(S^2-N^2)\pi^2} \quad (N \neq S,\ M, N, S \neq 0),\\
&N_{12}^{MNMS} = 0 \quad (M, N, S \neq 0), \quad N_{12}^{MNRN} = 0 \quad (M, R, N \neq 0),\\
&N_{12}^{M0MS} = 0 \quad (M, S \neq 0), \quad N_{12}^{MN0N} = 0 \quad (M, N \neq 0),\\
&N_{12}^{0NRS} = 0, \quad N_{12}^{MNR0} = 0,\\
&N_{13}^{MNRS} = \frac{2M\{1-(-1)^{M+R}\}\delta^{NS}g_{13}^*}{(M^2-R^2)\pi} \quad (M \neq R,\ M, R \neq 0),\\
&N_{13}^{NM0S} = \frac{2^{\frac{1}{2}}\{1-(-1)^M\}\delta^{NS}g_{13}^*}{M\pi} \quad (M \neq 0),\\
&N_{13}^{MNMS} = 0 \quad (M \neq 0), \quad N_{13}^{0NRS} = 0,\\
&N_{22}^{MNRS} = \delta^{RM}\delta^{SN}g_{22}^* \quad (N, S \neq 0),
\end{aligned}\right\} \quad (9.4j)$$

$$\left.\begin{aligned}
&N_{22}^{M0RS} = 0, \quad N_{22}^{MNR0} = 0,\\
&N_{23}^{MNRS} = \frac{2N\{1-(-1)^{N+S}\}\,\delta^{MR}\,g_{23}^*}{(N^2-S^2)\,\pi} \quad (N \neq S,\ N, S \neq 0),\\
&N_{23}^{MNR0} = \frac{2^{\frac{1}{2}}\{1-(-1)^{N}\}\,\delta^{MR}\,g_{23}^*}{N\pi} \quad (N \neq 0),\\
&N_{23}^{M0RS} = 0,\\
&N_{33}^{MNRS} = \delta^{MR}\,\delta^{NS}\,g_{33}^*,
\end{aligned}\right\} \qquad (9.4j,\ cont.)$$

$$\left.\begin{aligned}
&M_{11}^{MNRS} = \frac{4RN\{1-(-1)^{M+R}\}\{1-(-1)^{N+S}\}\,h_{11}^*}{(R^2-M^2)\,(N^2-S^2)\,\pi^2} \quad (M \neq R,\ N \neq S,\ M, R, N, S \neq 0),\\
&M_{11}^{MNR0} = \frac{2^{\frac{3}{2}}R\{1-(-1)^{M+R}\}\{1-(-1)^{N}\}\,h_{11}^*}{(R^2-M^2)\,N\pi^2} \quad (M \neq R,\ M, R, N \neq 0),\\
&M_{11}^{0NRS} = \frac{2^{\frac{3}{2}}N\{1-(-1)^{R}\}\{1-(-1)^{N+S}\}\,h_{11}^*}{R(N^2-S^2)\,\pi^2} \quad (N \neq S,\ R, N, S \neq 0),\\
&M_{11}^{0NR0} = \frac{2\{1-(-1)^{R}\}\{1-(-1)^{N}\}\,h_{11}^*}{RN\pi^2} \quad (N, R \neq 0),\\
&M_{11}^{MNMS} = 0 \quad (M, N, S \neq 0), \quad M_{11}^{MNRN} = 0 \quad (M, R, N \neq 0),\\
&M_{11}^{MNM0} = 0 \quad (M, N \neq 0), \quad M_{11}^{0NRN} = 0 \quad (N, R \neq 0),\\
&M_{11}^{M0RS} = 0, \quad M_{11}^{MN0S} = 0,\\
&M_{12}^{MNRS} = \delta^{MR}\,\delta^{NS}\,h_{12}^* \quad (N, S \neq 0),\\
&M_{12}^{M0RS} = 0, \quad M_{12}^{MNR0} = 0,\\
&M_{13}^{MNRS} = \frac{2N\{1-(-1)^{N+S}\}\,\delta^{MR}\,h_{13}^*}{(N^2-S^2)\,\pi} \quad (N \neq S,\ N, S \neq 0),\\
&M_{13}^{MNR0} = \frac{2^{\frac{1}{2}}\{1-(-1)^{N}\}\,\delta^{MR}\,h_{13}^*}{N\pi} \quad (N \neq 0),\\
&M_{13}^{MNRN} = 0 \quad (N \neq 0), \quad M_{13}^{M0RS} = 0,\\
&M_{21}^{MNRS} = \delta^{MR}\,\delta^{NS}\,h_{21}^* \quad (M, R \neq 1),\\
&M_{21}^{0NRS} = 0, \quad M_{21}^{MN0S} = 0,\\
&M_{22}^{MNRS} = \frac{4MS\{1-(-1)^{M+R}\}\{1-(-1)^{N+S}\}\,h_{22}^*}{(M^2-R^2)\,(S^2-N^2)\,\pi^2} \quad (M \neq R,\ N \neq S,\ M, R, N, S \neq 0),\\
&M_{22}^{MN0S} = \frac{2^{\frac{3}{2}}S\{1-(-1)^{M}\}\{1-(-1)^{N+S}\}\,h_{22}^*}{M(S^2-N^2)\,\pi^2} \quad (N \neq S,\ M, N, S \neq 0),\\
&M_{22}^{M0RS} = \frac{2^{\frac{3}{2}}M\{1-(-1)^{M+R}\}\{1-(-1)^{S}\}\,h_{22}^*}{(M^2-R^2)\,S\pi^2} \quad (M \neq R,\ M, R, S \neq 0),\\
&M_{22}^{M00S} = \frac{2\{1-(-1)^{M}\}\{1-(-1)^{S}\}\,h_{22}^*}{MS\pi^2} \quad (M, S \neq 0),
\end{aligned}\right\} \qquad (9.4k)$$

$$\left.\begin{aligned}
&M_{22}^{MNMS} = 0 \quad (M, N, S \neq 0), \quad M_{22}^{MNRN} = 0 \quad (M, R, N \neq 0),\\
&M_{22}^{MN0N} = 0 \quad (M, N \neq 0), \quad M_{22}^{M0MS} = 0 \quad (M, S \neq 0),\\
&M_{22}^{0NRS} = 0, \quad M_{22}^{MNR0} = 0,\\
&M_{23}^{MNRS} = \frac{2M\{1-(-1)^{M+R}\}\,\delta^{NS}\,h_{23}^{*}}{(M^2-R^2)\,\pi} \quad (M \neq R,\ M, R \neq 0),\\
&M_{23}^{MN0S} = \frac{2^{\frac{1}{2}}\{1-(-1)^{M}\}\,\delta^{NS}\,h_{23}^{*}}{M\pi} \quad (M \neq 0),\\
&M_{23}^{MNMS} = 0 \quad (M \neq 0), \quad M_{23}^{0NRS} = 0,\\
&M_{31}^{MNRS} = \frac{2N\{1-(-1)^{N+S}\}\,\delta^{MR}\,h_{31}^{*}}{(N^2-S^2)\,\pi} \quad (N \neq S,\ R, M, N, S \neq 0),\\
&M_{31}^{MNR0} = \frac{2^{\frac{1}{2}}\{1-(-1)^{N}\}\,\delta^{MR}\,h_{31}^{*}}{N\pi} \quad (M, R, N \neq 0),\\
&M_{31}^{MNRN} = 0 \quad (M, R, N \neq 0), \quad M_{31}^{M0RS} = 0, \quad M_{31}^{0NRS} = 0, \quad M_{31}^{MN0S} = 0,\\
&M_{32}^{MNRS} = \frac{2M\{1-(-1)^{M+R}\}\,\delta^{NS}\,h_{32}^{*}}{(M^2-R^2)\,\pi} \quad (M \neq R,\ M, R, N, S \neq 0),\\
&M_{32}^{MN0S} = \frac{2^{\frac{1}{2}}\{1-(-1)^{M}\}\,\delta^{NS}\,h_{32}^{*}}{M\pi} \quad (M, N, S \neq 0),\\
&M_{32}^{MNMS} = 0 \quad (M, N, S \neq 0), \quad M_{32}^{0NRS} = 0, \quad M_{32}^{M0RS} = 0, \quad M_{32}^{MNR0} = 0,\\
&M_{33}^{MNRS} = \frac{4MN\{1-(-1)^{M+R}\}\{1-(-1)^{N+S}\}\,h_{33}^{*}}{(M^2-R^2)\,(N^2-S^2)\,\pi^2} \quad (M \neq R,\ N \neq S,\ M, R, N, S \neq 0),\\
&M_{33}^{MNR0} = \frac{2^{\frac{3}{2}}M\{1-(-1)^{M+R}\}\{1-(-1)^{N}\}\,h_{33}^{*}}{(M^2-R^2)\,N\pi^2} \quad (M \neq R,\ M, R, N \neq 0),\\
&M_{33}^{MN0S} = \frac{2^{\frac{3}{2}}N\{1-(-1)^{M}\}\{1-(-1)^{N+S}\}\,h_{33}^{*}}{M(N^2-S^2)\,\pi^2} \quad (N \neq S,\ M, N, S \neq 0),\\
&M_{33}^{MN00} = \frac{2\{1-(-1)^{M}\}\{1-(-1)^{N}\}\,h_{33}^{*}}{MN\pi^2} \quad (M, N \neq 0),\\
&M_{33}^{MNMS} = 0 \quad (M, N, S \neq 0), \quad M_{33}^{MNRN} = 0 \quad (M, R, N \neq 0),\\
&M_{33}^{MNM0} = 0 \quad (M, N \neq 0), \quad M_{33}^{MN0N} = 0 \quad (M, N \neq 0),\\
&M_{33}^{0NRS} = 0, \quad M_{33}^{M0RS} = 0.
\end{aligned}\right\} \tag{9.4k, cont.}$$

In view of (9.2), it follows from (B 23) and (B 24) that

$$D'_{MN} = 0, \quad H'_{MNi} = 0 \tag{9.5}$$

and the electromagnetic equations (5.10)–(5.13) reduce to

$$\left.\begin{aligned}
-B'_{MN} - (M\pi/a)\,B_{MN1} - (N\pi/b)\,B_{MN2} &= \partial B_{MN3}/\partial\zeta,\\
E_{MN} + (M\pi/a)\,\overline{D}_{MN1} + (N\pi/b)\,\overline{D}_{MN2} &= \partial\overline{D}_{MN3}/\partial\zeta,
\end{aligned}\right\} \tag{9.6}$$

$$\left.\begin{aligned}\dot{B}_{MN1} &= \partial E_{MN2}/\partial\zeta + E'_{MN1} - (N\pi/b)\,E_{MN3},\\ \dot{B}_{MN2} &= -\partial E_{MN1}/\partial\zeta + E'_{MN2} + (M\pi/a)\,E_{MN3},\\ \dot{B}_{MN3} &= E'_{MN3} - (M\pi/a)\,E_{MN2} + (N\pi/b)\,E_{MN1},\end{aligned}\right\} \tag{9.7}$$

$$\left.\begin{aligned}-\dot{\bar{D}}_{MN1} &= J_{MN1} + \partial H_{MN2}/\partial\zeta + (N\pi/b)\,H_{MN3},\\ -\dot{\bar{D}}_{MN2} &= J_{MN2} - \partial H_{MN1}/\partial\zeta - (M\pi/a)\,H_{MN3},\\ -\dot{\bar{D}}_{MN3} &= J_{MN3} + (M\pi/a)\,H_{MN2} - (N\pi/b)\,H_{MN1}.\end{aligned}\right\} \tag{9.8}$$

The above interpretation of the electromagnetic part of the theory is particularly appropriate for problems such as an elastic rectangular wave guide in which the surface of the rod acts as a perfect conductor so that

$$E'_{MNi} = 0, \quad B'_{MN} = 0, \quad J_{MNi} = 0, \quad E_{MN} = 0. \tag{9.9}$$

To discuss the propagation of waves in a general anisotropic elastic wave guide it is necessary to limit the number of variables used to represent the electromagnetic fields. We choose

$$\left.\begin{aligned}&\tilde{E}_{011},\ \tilde{E}_{102},\ \tilde{E}_{111},\ \tilde{E}_{112},\ \tilde{E}_{113},\ \bar{D}_{011},\ \bar{D}_{102},\ \bar{D}_{111},\ \bar{D}_{112},\ \bar{D}_{113},\\ &\tilde{H}_{101},\ \tilde{H}_{012},\ \tilde{H}_{003},\ \tilde{H}_{103},\ \tilde{H}_{013},\ \tilde{H}_{111},\ \tilde{H}_{112},\ \tilde{H}_{113},\\ &B_{101},\ B_{012},\ B_{003},\ B_{103},\ B_{013},\ B_{111},\ B_{112},\ B_{113},\end{aligned}\right\} \tag{9.10}$$

with corresponding electromagnetic field equations in (9.7) and (9.8). Then, with these equations, with the mechanical equations of motion (7.2) and with the constitutive equations (6.26) and (9.4), discussion of isothermal wave propagation in the elastic wave guide is a straightforward but lengthy procedure. We do not consider this in detail, but note only one special case of a rigid isotropic wave guide. For this case it is possible to retain all the electromagnetic vectors with constitutive equations

$$\bar{D}_{MNi} = \epsilon\tilde{E}_{MNi}, \quad B_{MNi} = \mu\tilde{H}_{MNi}, \tag{9.11}$$

where ϵ, μ are the isotropic coefficients. Then, from (9.7), (9.8) and (9.11), the frequencies ω of wave propagation are given by

$$\mu\epsilon\omega^2 = k^2 + M^2\pi^2/a^2 + N^2\pi^2/b^2, \tag{9.12}$$

where $2\pi/k$ is the wave length and $M, N = 0, 1, \ldots$. The result (9.12) is, of course, the well known exact solution found directly from the three-dimensional equations. If the number of electromagnetic variables in the one-dimensional theory is limited to those in (9.10), then we recover (9.12) in the isotropic case for the values $M, N = 1, 0;\ 0, 1;\ 1, 1$.

10. Piezoelectric rod: an alternative formulation for rectangular cross-sections

For rods with rectangular sections it may be more convenient to deduce the isothermal piezoelectric equations from the theory of §9, rather than use the piezoelectric equations of §8. If the restricted mechanical theory of §6 is used here, and with magnetic fields absent from all constitutive equations in (6.26), we again have $B_{MNi} = 0$. Let ϕ be the applied potential at the surface of the rod and define

$$\phi_{MN} = \oint \phi \psi^M(\zeta^1)\, \bar{\chi}^N(\zeta^2)\, d\zeta^1, \quad \bar{\phi}_{MN} = \oint \phi \chi^M(\zeta^1)\, \bar{\psi}^N(\zeta^2)\, d\zeta^2, \tag{10.1}$$

where ψ^M, $\bar{\psi}^N$, χ^M, $\bar{\chi}^N$ are given by (9.2). Equations (9.3 a, c, d, f, g) still hold and the relevant electromagnetic field equations reduce to

$$\left.\begin{aligned} E_{0N1} &= -\bar{\phi}_{0N}, \quad E_{M02} = \phi_{M0}, \\ E_{MN2} &= \frac{Na}{Mb}(E_{MN1} + \bar{\phi}_{MN}) + \phi_{MN} \quad (M \neq 0), \\ E_{MN3} &= \frac{a}{M\pi}\frac{\partial}{\partial \zeta}(E_{MN1} + \bar{\phi}_{MN}) \quad (M \neq 0,\ N \neq 0) \end{aligned}\right\} \tag{10.2}$$

and

$$\frac{\partial \bar{D}_{MN3}}{\partial \zeta} = \frac{M\pi}{a}\bar{D}_{MN1} + \frac{N\pi}{b}\bar{D}_{MN2} \quad (M, N = 1, 2, \ldots). \tag{10.3}$$

The constitutive equations are given by (6.26), part of (6.27) and (9.4). Since, in a given problem, ϕ_{MN} and $\bar{\phi}_{MN}$ are specified, the only unknowns in (10.2) are E_{MN1} $(M, N = 1, 2, \ldots)$. With the constitutive equations, these unknowns are determined by equations (10.3), together with suitable initial and boundary conditions. In such piezoelectric problems we usually need to know the surface values of $\bar{D}^i$. These may be found from the representations

$$\left.\begin{aligned} \bar{D}^1 &= \sum_{M=0,\, N=1} \bar{D}_{MN1}\, \chi^M(\zeta^1)\, \bar{\psi}^N(\zeta^2), \\ \bar{D}^2 &= \sum_{M=1,\, N=0} \bar{D}_{MN2}\, \psi^M(\zeta^1)\, \bar{\chi}^N(\zeta^2), \\ \bar{D}^3 &= \sum_{M=1,\, N=1} \bar{D}_{MN3}\, \psi^M(\zeta^1)\, \bar{\psi}^N(\zeta^2). \end{aligned}\right\} \tag{10.4}$$

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Appendix A

This appendix contains a brief summary of the three-dimensional theory of electromagnetism of moving deformable media. In particular, consequences of all local field equations which result from the conservation laws are recorded in order to provide some background information for some aspects of the developments in §§3 and 4 of the paper.

Consider a body $\mathscr{B}$ consisting of particles X, and let $\mathscr{B}_t$ be the configuration of $\mathscr{B}$ at time t and $\mathscr{B}_R$ a reference configuration. Points in the reference configuration are specified by the position vector $\boldsymbol{X}^*$. A motion of $\mathscr{B}$ is defined by a sufficiently smooth vector function $\boldsymbol{\chi}^*$ which assigns to each X the place $\boldsymbol{x}^* = \boldsymbol{\chi}^*(\boldsymbol{X}^*, t)$ in the configuration $\mathscr{B}_t$ and $\boldsymbol{v}^* = \dot{\boldsymbol{\chi}}^*(\boldsymbol{X}^*, t)$ is velocity. The spatial forms of the electrodynamic and thermodynamical field equations in the three-dimensional theory are:

$$\operatorname{curl}^* \boldsymbol{e}^* = -(\dot{\boldsymbol{b}} + \boldsymbol{b}\operatorname{div}^* \boldsymbol{v}^* - \boldsymbol{L}^*\boldsymbol{b}), \quad \operatorname{div}^* \boldsymbol{b} = 0, \tag{A 1a, b}$$

$$\operatorname{curl}^* \boldsymbol{h}^* = \boldsymbol{j}^* + \dot{\bar{\boldsymbol{d}}} + \bar{\boldsymbol{d}}\operatorname{div}^* \boldsymbol{v}^* - \boldsymbol{L}^*\bar{\boldsymbol{d}}, \quad \operatorname{div}\bar{\boldsymbol{d}} = e, \tag{A 1c, d}$$

$$\dot{\rho}^* + \rho^* \operatorname{div}^* \boldsymbol{v}^* = 0,$$

$$\left.\begin{aligned} \rho^*\dot{\boldsymbol{v}}^* &= \rho^*(\boldsymbol{f}^* + \boldsymbol{f}_{\mathrm{e}}^*) + \operatorname{div}^* \boldsymbol{T}, \quad \boldsymbol{t} = \boldsymbol{T}\boldsymbol{u}, \\ \rho^*\boldsymbol{\Gamma}_{\mathrm{e}}^* &+ \boldsymbol{T} - \boldsymbol{T}^{\mathrm{T}} = \boldsymbol{0}, \\ \rho^*\dot{\eta}^* &= \rho^*(s^* + \xi^*) - \operatorname{div}\boldsymbol{p}^*, \quad k^* = \boldsymbol{p}^*\cdot\boldsymbol{u}, \quad h^* = \boldsymbol{q}^*\cdot\boldsymbol{u}, \quad \boldsymbol{q}^* = \theta^*\boldsymbol{p}^*, \\ \rho^* r^* &- \operatorname{div}^* \boldsymbol{q}^* - \rho^*\dot{\epsilon}^* + \rho^* w_{\mathrm{e}}^* + \boldsymbol{T}\cdot\boldsymbol{L}^* + \tfrac{1}{2}\rho^*\boldsymbol{\Gamma}_{\mathrm{e}}^*\cdot\boldsymbol{L}^* = 0, \end{aligned}\right\} \tag{A 2}$$

where div*, curl* are the divergence and curl operators with respect to the place $\boldsymbol{x}^*$ and

$$\boldsymbol{L}^* = \partial\boldsymbol{v}^*/\partial\boldsymbol{x}^*, \quad \boldsymbol{\Gamma}_{\mathrm{e}}^*\boldsymbol{z} = \boldsymbol{c}_{\mathrm{e}}^* \times \boldsymbol{z} \tag{A 3}$$

for every vector $\boldsymbol{z}$. Also, in (A 1) and (A 2) the temperature is denoted by θ^* (> 0), ρ^* is density, $\boldsymbol{f}^*$ is external body force density, $\boldsymbol{f}_{\mathrm{e}}^*$ is body force density, $\boldsymbol{c}_{\mathrm{e}}^*$ is body force couple density due to the electromagnetic field, $\boldsymbol{t}$ is surface traction across a surface in the configuration at time t whose unit outward normal is $\boldsymbol{u}$, h^* is flux of heat, k^* is flux of entropy, ϵ^* is internal energy density, η^* is entropy density, r^* is external volume rate of supply of heat density, s^* is external volume rate of supply of entropy density, w_{e}^* is volume rate of supply of electromagnetic energy density due to the electromagnetic fields, $\boldsymbol{e}$ is electric field vector, $\bar{\boldsymbol{d}}$ is electric displacement vector, $\boldsymbol{h}$ is magnetic field (axial) vector, $\boldsymbol{j}$ is current density, e is free charge, and

$$\boldsymbol{e}^* = \boldsymbol{e} + \boldsymbol{v}^* \times \boldsymbol{b}, \quad \boldsymbol{h}^* = \boldsymbol{h} - \boldsymbol{v} \times \bar{\boldsymbol{d}}, \quad \boldsymbol{j}^* = \boldsymbol{j} - e\boldsymbol{v}^*. \tag{A 4}$$

The symbol $\bar{\boldsymbol{d}}$ is used instead of $\boldsymbol{d}$ to avoid confusion with the notation for directors.

Corresponding field equations in material form are

$$\left.\begin{aligned} \operatorname{Curl}^* \boldsymbol{E} = -\dot{\boldsymbol{B}}, \quad \operatorname{Curl}^* \boldsymbol{H} &= \dot{\bar{\boldsymbol{D}}} + \boldsymbol{J}, \quad \operatorname{Div}^* \boldsymbol{B} = 0, \quad \operatorname{Div}^* \bar{\boldsymbol{D}} = E, \\ \rho_{\mathrm{R}}\dot{\boldsymbol{v}}^* &= \rho_{\mathrm{R}}(\boldsymbol{f}^* + \boldsymbol{f}_{\mathrm{e}}^*) + \operatorname{Div}^* \boldsymbol{T}_{\mathrm{R}}, \quad \boldsymbol{t}_{\mathrm{R}} = \boldsymbol{T}_{\mathrm{R}}\boldsymbol{u}_{\mathrm{R}}, \\ \rho_{\mathrm{R}}\boldsymbol{\Gamma}_{\mathrm{e}}^* &+ \boldsymbol{T}_{\mathrm{R}}\boldsymbol{F}^{*\mathrm{T}} - \boldsymbol{F}^*\boldsymbol{T}_{\mathrm{R}}^{\mathrm{T}} = \boldsymbol{0}, \\ \rho_{\mathrm{R}}\dot{\eta}^* &= \rho_{\mathrm{R}}(s^* + \xi^*) - \operatorname{Div}^* \boldsymbol{p}_{\mathrm{R}}^*, \\ k_{\mathrm{R}}^* = \boldsymbol{p}_{\mathrm{R}}^*\cdot\boldsymbol{u}_{\mathrm{R}}, \quad h_{\mathrm{R}}^* &= \boldsymbol{q}_{\mathrm{R}}^*\cdot\boldsymbol{u}_{\mathrm{R}}, \quad \boldsymbol{q}_{\mathrm{R}}^* = \theta^*\boldsymbol{p}_{\mathrm{R}}^*, \\ \rho_{\mathrm{R}}^* r^* - \operatorname{Div}^* \boldsymbol{q}_{\mathrm{R}}^* &- \rho_{\mathrm{R}}^*\dot{\epsilon}^* + \rho_{\mathrm{R}}^* w_{\mathrm{e}}^* + \boldsymbol{T}_{\mathrm{R}}\cdot\dot{\boldsymbol{F}}^* + \tfrac{1}{2}\rho_{\mathrm{R}}^*\boldsymbol{\Gamma}_{\mathrm{e}}^*\cdot\boldsymbol{L}^* = 0, \end{aligned}\right\} \tag{A 5}$$

where Div*, Curl* are divergence and curl operators with respect to $\boldsymbol{X}^*$, ρ_{R}^* is reference density, $\boldsymbol{t}_{\mathrm{R}}$ is surface traction across a surface in the configuration at time t measured per unit area of the corresponding surface in the reference configuration whose unit outward normal is $\boldsymbol{u}_{\mathrm{R}}$, h_{R}^* is flux of heat and k_{R}^* is flux of entropy, measured per unit area of the reference surface, and

$$\left.\begin{aligned} \boldsymbol{F}^* = \partial\boldsymbol{\chi}^*/\partial\boldsymbol{X}^*, \quad \rho_{\mathrm{R}}^* &= \Gamma^*\rho^*, \quad \Gamma^* = \det\boldsymbol{F}^* > 0, \\ \Gamma^*\boldsymbol{T} = \boldsymbol{T}_{\mathrm{R}}\boldsymbol{F}^{*\mathrm{T}}, \quad \Gamma^*\boldsymbol{p}^* &= \boldsymbol{F}^*\boldsymbol{p}_{\mathrm{R}}^*, \quad \Gamma^*\boldsymbol{q}^* = \boldsymbol{F}^*\boldsymbol{q}_{\mathrm{R}}, \\ \boldsymbol{E} = \boldsymbol{F}^{*\mathrm{T}}\boldsymbol{e}^*, \quad \boldsymbol{H} = \boldsymbol{F}^{*\mathrm{T}}\boldsymbol{h}^*, \quad \bar{\boldsymbol{D}} &= \Gamma^*\boldsymbol{F}^{*-1}\bar{\boldsymbol{d}}, \quad \boldsymbol{B} = \Gamma^*\boldsymbol{F}^{*-1}\boldsymbol{b}, \\ E = \Gamma^* e, \quad \boldsymbol{J} &= \Gamma^*\boldsymbol{F}^{*-1}\boldsymbol{j}^*. \end{aligned}\right\} \tag{A 6}$$

In the development of rod theory from three-dimensional equations, it is convenient to introduce ζ^i $(i = 1, 2, 3)$, with $\zeta^3 = \zeta$, as a system of curvilinear coordinates in the reference configuration $\mathscr{B}_R$ and to use these coordinates as a convected system for the body in its configuration $\mathscr{B}_t$. Then

$$\left.\begin{aligned} &\boldsymbol{X}^* = \boldsymbol{X}^*(\zeta^i), \quad \boldsymbol{x}^* = \boldsymbol{x}^*(\zeta^i, t),\\ &\boldsymbol{G}_i = \partial \boldsymbol{X}^*/\partial \zeta^i, \quad \boldsymbol{g}_i = \partial \boldsymbol{x}^*/\partial \zeta^i,\\ &\boldsymbol{G}^i \cdot \boldsymbol{G}_j = \delta^i_j, \quad \boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta^i_j, \quad G_{ij} = \boldsymbol{G}_i \cdot \boldsymbol{G}_j, \quad G^{ij} = \boldsymbol{G}^i \cdot \boldsymbol{G}^j,\\ &g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j, \quad g^{ij} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j, \quad g^{\frac{1}{2}} = [\boldsymbol{g}_1 \boldsymbol{g}_2 \boldsymbol{g}_3], \quad G^{\frac{1}{2}} = [\boldsymbol{G}_1 \boldsymbol{G}_2 \boldsymbol{G}_3],\\ &\boldsymbol{F}^* = \boldsymbol{g}_i \otimes \boldsymbol{G}^i, \quad \boldsymbol{g}_i = \boldsymbol{F}^* \boldsymbol{G}_i, \quad \boldsymbol{G}^i = \boldsymbol{F}^{*\mathrm{T}} \boldsymbol{g}^i,\\ &\Gamma^* = \det \boldsymbol{F}^* = g^{\frac{1}{2}}/G^{\frac{1}{2}}, \quad \boldsymbol{L}^* = \dot{\boldsymbol{g}}_i \otimes \boldsymbol{g}^i, \end{aligned}\right\} \tag{A 7}$$

where $\boldsymbol{g}_i$, $\boldsymbol{g}^i$ are covariant and contravariant base vectors, respectively, g_{ij}, g^{ij} are covariant and contravariant metric tensors, respectively, in the configuration $\mathscr{B}_t$, and δ^i_j is the Kronecker delta. Corresponding quantities for the reference configuration are $\boldsymbol{G}_i$, $\boldsymbol{G}^i$, G_{ij}, G^{ij}. Also,

$$\left.\begin{aligned} &\boldsymbol{u} = u_i \boldsymbol{g}^i = u^i \boldsymbol{g}_i, \quad \boldsymbol{u}_R = u_{Ri} \boldsymbol{G}^i = u^i_R \boldsymbol{G}_i, \quad \mu = \rho^* g^{\frac{1}{2}} = \rho^*_R G^{\frac{1}{2}},\\ &\boldsymbol{t}^i = \boldsymbol{T} \boldsymbol{g}^i, \quad \boldsymbol{t}^i_R = \boldsymbol{T}_R \boldsymbol{G}^i, \quad \boldsymbol{T}^i = g^{\frac{1}{2}} \boldsymbol{t}^i = G^{\frac{1}{2}} \boldsymbol{t}^i_R, \quad \boldsymbol{T} = \boldsymbol{t}^i \otimes \boldsymbol{g}_i, \quad \boldsymbol{T}_R = \boldsymbol{t}^i_R \otimes \boldsymbol{G}_i,\\ &g^{\frac{1}{2}} \operatorname{div}^* \boldsymbol{T} = G^{\frac{1}{2}} \operatorname{Div}^* \boldsymbol{T}_R = \boldsymbol{T}^i{}_{,i},\\ &\boldsymbol{p}^* = p^{*i} \boldsymbol{g}_i, \quad \boldsymbol{p}^*_R = p^{*i}_R \boldsymbol{G}_i, \quad P^i = g^{\frac{1}{2}} p^{*i} = G^{\frac{1}{2}} p^{*i}_R,\\ &g^{\frac{1}{2}} \operatorname{div} \boldsymbol{p}^* = G^{\frac{1}{2}} \operatorname{Div} \boldsymbol{p}^*_R = P^i{}_{,i}, \end{aligned}\right\} \tag{A 8}$$

where a comma denotes partial differentiation with respect to ζ^i. The electromagnetic vectors, when referred to various bases $\boldsymbol{g}^i$, $\boldsymbol{G}^i$, etc. can be expressed in terms of their components as

$$\left.\begin{aligned} &\boldsymbol{e}^* = e^*_i \boldsymbol{g}^i, \quad \boldsymbol{h}^* = h^*_i \boldsymbol{g}^i, \quad \boldsymbol{E} = E_i \boldsymbol{G}^i, \quad \boldsymbol{H} = H_i \boldsymbol{G}^i,\\ &\bar{\boldsymbol{d}} = \bar{d}^i \boldsymbol{g}_i, \quad \boldsymbol{b} = b^i \boldsymbol{g}_i, \quad \bar{\boldsymbol{D}} = \bar{D}^i \boldsymbol{G}_i, \quad \boldsymbol{B} = B^i \boldsymbol{G}_i,\\ &\boldsymbol{j}^* = j^{*i} \boldsymbol{g}_i, \quad \boldsymbol{J} = J^i \boldsymbol{G}_i, \end{aligned}\right\} \tag{A 9}$$

so that, with the help of (A 6), we have

$$\left.\begin{aligned} &E_i = e^*_i, \quad H_i = h^*_i, \quad \hat{D}^i = G^{\frac{1}{2}} \bar{D}^i = g^{\frac{1}{2}} \bar{d}^i, \quad \hat{B}^i = G^{\frac{1}{2}} B^i = g^{\frac{1}{2}} b^i,\\ &\hat{J}^i = G^{\frac{1}{2}} J^i = g^{\frac{1}{2}} j^{*i}, \quad \hat{E} = G^{\frac{1}{2}} E = g^{\frac{1}{2}} e. \end{aligned}\right\} \tag{A 10}$$

Finally, we need values for $\boldsymbol{\Gamma}^*_e$, $\boldsymbol{c}^*_e$, $\boldsymbol{f}^*_e$ and w^*_e used previously by Green & Naghdi (1983), which were a slight modification of those derived by Hutter and van der Ven (1978). Thus,

$$\left.\begin{aligned} &\rho^* \boldsymbol{\Gamma}^*_e = \boldsymbol{T}_e - \boldsymbol{T}^{\mathrm{T}}_e, \quad \rho^* \boldsymbol{c}^*_e = \boldsymbol{g}_i \times \boldsymbol{t}^i_e, \quad \boldsymbol{t}^i_e = \boldsymbol{T}_e \boldsymbol{g}^i,\\ &\boldsymbol{T}_e = \boldsymbol{e}^* \otimes \bar{\boldsymbol{d}} + \boldsymbol{h}^* \otimes \boldsymbol{b} - \tfrac{1}{2}(\epsilon_0 \boldsymbol{e}^* \cdot \boldsymbol{e}^* + \mu_0 \boldsymbol{h}^* \cdot \boldsymbol{h}^*) \boldsymbol{I},\\ &\rho^* \boldsymbol{f}^*_e = e \boldsymbol{e}^* + \boldsymbol{j}^* \times \boldsymbol{b} + (\bar{\boldsymbol{d}} - \epsilon_0 \boldsymbol{e}^*) \cdot \nabla \boldsymbol{e}^* + (\boldsymbol{b} - \mu_0 \boldsymbol{h}^*) \cdot \nabla \boldsymbol{h}^*\\ &\qquad + \overline{\dot{\bar{\boldsymbol{d}}} \times \boldsymbol{b}} + (\bar{\boldsymbol{d}} \times \boldsymbol{b}) \operatorname{div}^* \boldsymbol{v}^* + \boldsymbol{L}^{*\mathrm{T}} (\bar{\boldsymbol{d}} \times \boldsymbol{b}),\\ &\rho^* w^*_e + \tfrac{1}{2} \rho^* \boldsymbol{\Gamma}^*_e \cdot \boldsymbol{L}^* = \boldsymbol{T}_e \cdot \boldsymbol{L}^* + \boldsymbol{e}^* \cdot \boldsymbol{j}^* + \boldsymbol{e}^* \cdot (\dot{\bar{\boldsymbol{d}}} + \bar{\boldsymbol{d}} \operatorname{div}^* \boldsymbol{v}^* - \boldsymbol{L}^* \bar{\boldsymbol{d}})\\ &\qquad + \boldsymbol{h}^* \cdot (\dot{\boldsymbol{b}} + \boldsymbol{b} \operatorname{div}^* \boldsymbol{v}^* - \boldsymbol{L}^* \boldsymbol{b}),\\ &\rho^*_R w^*_e + \tfrac{1}{2} \rho^*_R \boldsymbol{\Gamma}^*_e \cdot \boldsymbol{L}^* = \boldsymbol{T}_{Re} \cdot \dot{\boldsymbol{F}}^* + \boldsymbol{E} \cdot \boldsymbol{J} + \boldsymbol{E} \cdot \dot{\bar{\boldsymbol{D}}} + \boldsymbol{H} \cdot \dot{\boldsymbol{B}}, \quad \Gamma^* \boldsymbol{T}_e = \boldsymbol{T}_{Re} \boldsymbol{F}^{*\mathrm{T}}, \end{aligned}\right\} \tag{A 11}$$

where ϵ_0, μ_0 are the electromagnetic coefficients for vacuum.

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Appendix B

The purpose of this appendix is to provide some formulae which arise in the development of rod theory from three-dimensional equations given in Appendix A. Formulae of the type obtained here have been given previously in the context of thermomechanical theory (see Green & Naghdi 1979). However, in this appendix, we provide slightly more general formulae which include results from electromagnetism, and there is some change in notation. The mechanical theory is restricted to a Cosserat curve with two directors, but more generality is allowed in the thermal and electromagnetic effects. We suppose that position vectors and temperature in the reference and in the current configurations of the body are specified by

$$\left.\begin{aligned} &\boldsymbol{X}^* = \boldsymbol{R}+\lambda^\alpha(\zeta^1,\zeta^2)\,\boldsymbol{D}_\alpha, \quad \boldsymbol{R}=\boldsymbol{R}(\zeta), \quad \boldsymbol{D}_\alpha=\boldsymbol{D}_\alpha(\zeta), \quad \zeta^3=\zeta,\\ &\boldsymbol{x}^* = \boldsymbol{r}+\lambda^\alpha(\zeta^1,\zeta^2)\,\boldsymbol{d}_\alpha, \quad \boldsymbol{r}=\boldsymbol{r}(\zeta,t), \quad \boldsymbol{d}_\alpha=\boldsymbol{d}_\alpha(\zeta,t),\\ &\theta^* = \theta+\sum_{K=1}^{P}\mu^M(\zeta^1)\,\bar{\mu}^N(\zeta^2)\,\theta_{MN}, \quad \theta=\theta(\zeta,t), \quad \theta_{MN}=\theta_{MN}(\zeta,t),\\ &\mu^0=\bar{\mu}^0=1, \end{aligned}\right\} \tag{B 1}$$

in the region $\zeta_1 \leqslant \zeta \leqslant \zeta_2$, where Greek indices take the values 1 and 2, capital Latin indices M, N are integers or zeros, and $M+N=K$. The curve $\zeta^\alpha = 0$ is identified with the curve c in the theory of §2 and the rod occupies some neighbourhood of c bounded by the surface

$$F(\zeta^1,\zeta^2)=0, \tag{B 2}$$

which is such that $\zeta=$ constant are sections of the rod bounded by closed curves in this surface. In Green *et al.* (1974) and Green & Naghdi (1979), slightly more generality was allowed, by choosing $F(\zeta^1,\zeta^2,\zeta^3)=0$, but the form (B 2) is sufficient for our purpose.

Then, using the notations and definitions of §2, and Appendix A, we have

$$\left.\begin{aligned} \lambda &= \rho a_{33}^{\frac{1}{2}} = \rho_{\mathrm{R}} A_{33}^{\frac{1}{2}} = \iint \mu\,\mathrm{d}A, \quad \mathrm{d}A = \mathrm{d}\zeta^1\,\mathrm{d}\zeta^2,\\ \lambda y^{\alpha 0} &= \lambda y^{0\alpha} = \iint \lambda^\alpha(\zeta^1,\zeta^2)\,\mu\,\mathrm{d}A,\\ \lambda y^{\alpha\beta} &= \lambda y^{\beta\alpha} = \iint \lambda^\alpha(\zeta^1,\zeta^2)\,\lambda^\beta(\zeta^1,\zeta^2)\,\mu\,\mathrm{d}A, \end{aligned}\right\} \tag{B 3}$$

the double integrals being over the surface $\zeta=$ constant bounded by (B 2). Also

$$\left.\begin{aligned} \boldsymbol{n} &= \boldsymbol{n}^3 a_{33}^{\frac{1}{2}} = {}_{\mathrm{R}}\boldsymbol{n} A_{33}^{\frac{1}{2}} = \iint \boldsymbol{T}^3\,\mathrm{d}A,\\ \boldsymbol{m}^\alpha &= \boldsymbol{m}^{\alpha 3} a_{33}^{\frac{1}{2}} = {}_{\mathrm{R}}\boldsymbol{m}^{\alpha 3} A_{33}^{\frac{1}{2}} = \iint \boldsymbol{T}^3 \lambda^\alpha(\zeta^1,\zeta^2)\,\mathrm{d}A,\\ \boldsymbol{k}^\alpha &= \boldsymbol{k}^{\alpha 3} a_{33}^{\frac{1}{2}} = {}_{\mathrm{R}}\boldsymbol{k}^{\alpha 3} a_{33}^{\frac{1}{2}} = \iint \boldsymbol{T}^\beta \frac{\partial \lambda^\alpha(\zeta^1,\zeta^2)}{\partial \zeta^\beta}\,\mathrm{d}A, \end{aligned}\right\} \tag{B 4}$$

$$\left.\begin{aligned} &\lambda \boldsymbol{f} = \iint \mu \boldsymbol{f}^*\,\mathrm{d}A + \oint (\boldsymbol{T}^1\,\mathrm{d}\zeta^2 - \boldsymbol{T}^2\,\mathrm{d}\zeta^1), \quad \lambda \boldsymbol{f}_{\mathrm{e}} = \iint \mu \boldsymbol{f}_{\mathrm{e}}^*\,\mathrm{d}A,\\ &\lambda \boldsymbol{l}^\alpha = \iint \mu \boldsymbol{f}^* \lambda^\alpha(\zeta^1,\zeta^2)\,\mathrm{d}A + \oint \lambda^\alpha(\zeta^1,\zeta^2)\,(\boldsymbol{T}^1\,\mathrm{d}\zeta^2 - \boldsymbol{T}^2\,\mathrm{d}\zeta^1),\\ &\lambda \boldsymbol{l}_{\mathrm{e}}^\alpha = \iint \mu \boldsymbol{f}_{\mathrm{e}}^* \lambda^\alpha(\zeta^1,\zeta^2)\,\mathrm{d}A, \quad \lambda \boldsymbol{c}_{\mathrm{e}} = \iint \mu \boldsymbol{c}_{\mathrm{e}}^*\,\mathrm{d}A, \end{aligned}\right\} \tag{B 5}$$

$$\left.\begin{aligned}
\lambda\bar{w} &= \iint \mu(w_{\mathrm{e}}^{*}+\tfrac{1}{2}\boldsymbol{\Gamma}_{\mathrm{e}}^{*}\cdot\boldsymbol{L}^{*})\,\mathrm{d}A,\\
\lambda\eta &= \iint \mu\eta^{*}\,\mathrm{d}A, \quad \lambda\eta_{MN} = \iint \mu\eta^{*}\mu^{M}(\zeta^{1})\,\bar{\mu}^{N}(\zeta^{2})\,\mathrm{d}A,\\
\lambda s &= \iint \mu s^{*}\,\mathrm{d}A - \oint (P^{1}\,\mathrm{d}\zeta^{2}-P^{2}\,\mathrm{d}\zeta^{1}),\\
\lambda s_{MN} &= \iint \mu s^{*}\mu^{M}(\zeta^{1})\,\bar{\mu}^{N}(\zeta^{2})\,\mathrm{d}A - \oint \mu^{M}(\zeta^{1})\,\bar{\mu}^{N}(\zeta^{2})\,(P^{1}\,\mathrm{d}\zeta^{2}-P^{2}\,\mathrm{d}\zeta^{1}),\\
\lambda\xi &= \iint \mu\xi^{*}\,\mathrm{d}A,\\
\lambda\xi_{MN} &= \iint \mu\xi^{*}\mu^{M}(\zeta^{1})\,\bar{\mu}^{N}(\zeta^{2})\,\mathrm{d}A + \iint P^{\alpha}\frac{\partial\mu^{M}(\zeta^{1})\,\bar{\mu}^{N}(\zeta^{2})}{\partial\zeta^{\alpha}}\,\mathrm{d}A,\\
k &= \iint P^{3}\,\mathrm{d}A, \quad k_{MN} = \iint P^{3}\mu^{M}(\zeta^{1})\,\bar{\mu}^{N}(\zeta^{2})\,\mathrm{d}A,
\end{aligned}\right\} \quad \text{(B 6)}$$

where the line integrals are along the intersection of the surface (B 2) with ζ = constant.

Previously, in Green & Naghdi (1979), we have used powers of ζ^1, ζ^2 in our study of the thermomechanical equations, i.e.

$$\lambda^{\alpha}(\zeta^{1},\zeta^{2}) = \zeta^{\alpha}, \quad \mu^{M}(\zeta^{1}) = (\zeta^{1})^{M}, \quad \bar{\mu}^{N}(\zeta^{2}) = (\zeta^{2})^{N},$$

although the notational arrangement was different. For discussion of the electromagnetic equations it is convenient to introduce two other sets of functions $\chi^M(\zeta^1)$, $\bar{\chi}^N(\zeta^2)$, $\psi^M(\zeta^1)$, $\bar{\psi}^N(\zeta^2)$ with the properties that

$$\left.\begin{aligned}
\frac{\mathrm{d}\chi^{M}(\zeta^{1})}{\mathrm{d}\zeta^{1}} &= \sum_{K=0}^{M}\chi_{K}^{M}\,\psi^{K}(\zeta^{1}), \quad \frac{\mathrm{d}\psi^{M}(\zeta^{1})}{\mathrm{d}\zeta^{1}} = \sum_{K=0}^{M}\psi_{K}^{M}\,\chi^{K}(\zeta^{1}),\\
\frac{\mathrm{d}\bar{\chi}^{N}(\zeta^{2})}{\mathrm{d}\zeta^{2}} &= \sum_{K=0}^{N}\bar{\chi}_{K}^{N}\,\bar{\psi}^{K}(\zeta^{2}), \quad \frac{\mathrm{d}\bar{\psi}^{N}(\zeta^{2})}{\mathrm{d}\zeta^{2}} = \sum_{K=0}^{N}\bar{\psi}_{K}^{N}\,\bar{\chi}^{K}(\zeta^{2}).
\end{aligned}\right\} \quad \text{(B 7)}$$

We first consider the spatial forms (A 1) of the electromagnetic field equations. Multiplying (A 1*b*, *d*) by $\chi^M(\zeta^1)\,\bar{\chi}^N(\zeta^2)$ and $\psi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)$, respectively, and integrating over a material region $\mathscr{P}^*$ in the configuration at time t gives

$$\left.\begin{aligned}
\int_{\partial\mathscr{P}^{*}}\chi^{M}(\zeta^{1})\,\bar{\chi}^{N}(\zeta^{2})\,\boldsymbol{b}\cdot\mathrm{d}\boldsymbol{a} &= \int_{\mathscr{P}^{*}}\frac{\partial\chi^{M}(\zeta^{1})\,\bar{\chi}^{N}(\zeta^{2})}{\partial\zeta^{\alpha}}\,\boldsymbol{g}^{\alpha}\cdot\boldsymbol{b}\,\mathrm{d}v,\\
\int_{\partial\mathscr{P}^{*}}\psi^{M}(\zeta^{1})\,\bar{\psi}^{N}(\zeta^{2})\,\bar{\boldsymbol{d}}\cdot\mathrm{d}\boldsymbol{a} &= \int_{\mathscr{P}^{*}}\left\{e\psi^{M}(\zeta^{1})\,\bar{\psi}^{N}(\zeta^{2}) + \frac{\partial\psi^{M}(\zeta^{1})\,\bar{\psi}^{N}(\zeta^{2})}{\partial\zeta^{\alpha}}\,\boldsymbol{g}^{\alpha}\cdot\bar{\boldsymbol{d}}\right\}\mathrm{d}v.
\end{aligned}\right\} \quad \text{(B 8)}$$

Next we take the scalar product of (A 1*a*) with

$$\psi^{M}(\zeta^{1})\,\bar{\chi}^{N}(\zeta^{2})\,\boldsymbol{g}^{1}, \quad \chi^{M}(\zeta^{1})\,\bar{\psi}^{N}(\zeta^{2})\,\boldsymbol{g}^{2}, \quad \chi^{M}(\zeta^{1})\,\bar{\chi}^{N}(\zeta^{2})\,\boldsymbol{g}^{3}, \quad \text{(B 9)}$$

respectively, and integrate over $\mathscr{P}^*$. This yields the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*}\psi^M(\zeta^1)\,\bar{\chi}^N(\zeta^2)\,\boldsymbol{g}^1\cdot\boldsymbol{b}\,\mathrm{d}v=\int_{\partial\mathscr{P}^*}\psi^M(\zeta^1)\,\bar{\chi}^N(\zeta^2)\,[\boldsymbol{g}^1\times\boldsymbol{e}^*\cdot\mathrm{d}\boldsymbol{a}]+\int_{\mathscr{P}^*}\psi^M(\zeta^1)\,\frac{\mathrm{d}\bar{\chi}^N(\zeta^2)}{\mathrm{d}\zeta^2}\,[\boldsymbol{g}^1\boldsymbol{g}^2\boldsymbol{e}^*]\,\mathrm{d}v, \tag{B 10a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*}\chi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,\boldsymbol{g}^2\cdot\boldsymbol{b}\,\mathrm{d}v=\int_{\partial\mathscr{P}^*}\chi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,[\boldsymbol{g}^2\times\boldsymbol{e}^*\cdot\mathrm{d}\boldsymbol{a}]-\int_{\mathscr{P}^*}\frac{\mathrm{d}\chi^M(\zeta^1)}{\mathrm{d}\zeta^1}\,\bar{\psi}^N(\zeta^2)\,[\boldsymbol{g}^1\boldsymbol{g}^2\boldsymbol{e}^*]\,\mathrm{d}v, \tag{B 10b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*}\chi^M(\zeta^1)\,\bar{\chi}^N(\zeta^2)\,b^3\,\mathrm{d}v=\int_{\partial\mathscr{P}^*}\chi^M(\zeta^1)\,\bar{\chi}^N(\zeta^2)\,[\boldsymbol{g}^3\times\boldsymbol{e}^*\cdot\mathrm{d}\boldsymbol{a}]-\int_{\mathscr{P}^*}\left[\left\{\frac{\mathrm{d}\chi^M(\zeta^1)}{\mathrm{d}\zeta^1}\,\bar{\chi}^N(\zeta^2)\,\boldsymbol{g}^1+\chi^M(\zeta^1)\,\frac{\mathrm{d}\bar{\chi}^N(\zeta^2)}{\mathrm{d}\zeta^2}\,\boldsymbol{g}^2\right\}\times\boldsymbol{g}^3\cdot\boldsymbol{e}^*\right]\mathrm{d}v. \tag{B 10c}$$

Similarly, we take the scalar product of (A 1c) with

$$\chi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,\boldsymbol{g}^1,\quad \psi^M(\zeta^1)\,\bar{\chi}^N(\zeta^2)\,\boldsymbol{g}^2,\quad \psi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,\boldsymbol{g}^3, \tag{B 11}$$

respectively, and integrate over $\mathscr{P}^*$. Thus, we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*}\chi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,\boldsymbol{g}^1\cdot\bar{\boldsymbol{d}}\,\mathrm{d}v=\int_{\partial\mathscr{P}^*}\chi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,[\boldsymbol{g}^1\times\boldsymbol{h}^*\cdot\mathrm{d}\boldsymbol{a}]+\int_{\mathscr{P}^*}\chi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,\boldsymbol{g}^1\cdot\boldsymbol{j}^*\,\mathrm{d}v+\int_{\mathscr{P}^*}\chi^M(\zeta^1)\,\frac{\mathrm{d}\bar{\psi}^N(\zeta^2)}{\mathrm{d}\zeta^2}\,[\boldsymbol{g}^1\boldsymbol{g}^2\boldsymbol{h}^*]\,\mathrm{d}v, \tag{B 12a}$$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*}\psi^M(\zeta^1)\,\bar{\chi}^N(\zeta^2)\,\boldsymbol{g}^2\cdot\bar{\boldsymbol{d}}\,\mathrm{d}v=\int_{\partial\mathscr{P}^*}\psi^M(\zeta^1)\,\bar{\chi}^N(\zeta^2)\,[\boldsymbol{g}^2\times\boldsymbol{h}^*\cdot\mathrm{d}\boldsymbol{a}]+\int_{\mathscr{P}^*}\psi^M(\zeta^1)\,\bar{\chi}^N(\zeta^2)\,\boldsymbol{g}^2\cdot\boldsymbol{j}^*\,\mathrm{d}v-\int_{\mathscr{P}^*}\frac{\mathrm{d}\psi^M(\zeta^1)}{\mathrm{d}\zeta^1}\,\bar{\chi}^N(\zeta^2)\,[\boldsymbol{g}^1\boldsymbol{g}^2\boldsymbol{h}^*]\,\mathrm{d}v, \tag{B 12b}$$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*}\psi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,\bar{d}^3\,\mathrm{d}v=\int_{\partial\mathscr{P}^*}\psi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,[\boldsymbol{g}^3\times\boldsymbol{h}^*\cdot\mathrm{d}\boldsymbol{a}]+\int_{\mathscr{P}^*}\psi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,\boldsymbol{j}^*\cdot\boldsymbol{g}^3\,\mathrm{d}v-\int_{\mathscr{P}^*}\left[\left\{\frac{\mathrm{d}\psi^M(\zeta^1)}{\mathrm{d}\zeta^1}\,\bar{\psi}^N(\zeta^2)\,\boldsymbol{g}^1+\psi^M(\zeta^1)\,\frac{\mathrm{d}\bar{\psi}^N(\zeta^2)}{\mathrm{d}\zeta^2}\,\boldsymbol{g}^2\right\}\times\boldsymbol{g}^3\cdot\boldsymbol{h}^*\right]\mathrm{d}v. \tag{B 12c}$$

Similarly, from the material equations (A 5) we may obtain equations of the same form as those in (B 8)–(B 12) if we replace $\mathscr{P}^*$, $\partial\mathscr{P}^*$, dv, $d\boldsymbol{a}$, $\boldsymbol{e}^*$, $\boldsymbol{h}^*$, e, b^i, $\bar{d}^i$, $\boldsymbol{a}_i$, $\boldsymbol{g}^i$ in these equations by ${}_{\mathrm{R}}\mathscr{P}^*$, $\partial_{\mathrm{R}}\mathscr{P}^*$, dV, dA, $\boldsymbol{E}$, $\boldsymbol{H}$, E, B^i, $\bar{D}^i$, $\boldsymbol{A}_i$, $\boldsymbol{G}^i$, respectively.

The equations (B 8), (B 10) and (B 12), and their material counterparts, are now applied to a rod-like region bounded by the surface (B 2) and the surfaces $\zeta = \zeta_1$, $\zeta = \zeta_2$. The resulting integrals are along the curve $\zeta^\alpha = 0$ bounded by the ends $\zeta = \zeta_1$, $\zeta = \zeta_2$. The integral balances in spatial form are

$$\int_{\zeta_1}^{\zeta_2} b'_{MN}\,ds + [\boldsymbol{b}_{MN}\cdot\boldsymbol{v}]_{\zeta_1}^{\zeta_2} = \int_{\zeta_1}^{\zeta_2}\left(\sum_{K=0}^{M}\chi_K^M\, b_{KN}{}^1 + \sum_{K=0}^{N}\bar{\chi}_K^N\, b_{MN}{}^2\right) ds, \tag{B 13}$$

$$\int_{\zeta_1}^{\zeta_2} d'_{MN}\,ds + [\bar{\boldsymbol{d}}_{MN}\cdot\boldsymbol{v}]_{\zeta_1}^{\zeta_2} = \int_{\zeta_1}^{\zeta_2}\left(e_{MN} + \sum_{K=0}^{M}\psi_K^M\, \bar{d}_{KN}{}^1 + \sum_{K=0}^{N}\bar{\psi}_K^N\, \bar{d}_{MK}{}^2\right) ds, \tag{B 14}$$

$$\left.\begin{aligned}
\frac{d}{dt}\int_{\zeta_1}^{\zeta_2} b_{MN}{}^1\,ds &= [\boldsymbol{a}^1\cdot\boldsymbol{e}^*_{MN}\times\boldsymbol{v}]_{\zeta_1}^{\zeta_2} + \int_{\zeta_1}^{\zeta_2}\left(a_{33}^{\frac{1}{2}}\, e'_{MN}{}^1 + \sum_{K=0}^{N}\bar{\chi}_K^N\, e^*_{MK3}\right) d\zeta,\\
\frac{d}{dt}\int_{\zeta_1}^{\zeta_2} b_{MN}{}^2\,ds &= [\boldsymbol{a}^2\cdot\boldsymbol{e}^*_{MN}\times\boldsymbol{v}]_{\zeta_1}^{\zeta_2} + \int_{\zeta_1}^{\zeta_2}\left(a_{33}^{\frac{1}{2}}\, e'_{MN}{}^2 - \sum_{K=0}^{M}\chi_K^M\, e^*_{KN3}\right) d\zeta,\\
\frac{d}{dt}\int_{\zeta_1}^{\zeta_2} b_{MN}{}^3\,ds &= \int_{\zeta_1}^{\zeta_2}\left(a_{33}^{\frac{1}{2}}\, e'_{MN}{}^3 + \sum_{K=0}^{M}\chi_K^M\, e^*_{KN2} - \sum_{K=0}^{N}\bar{\chi}_K^N\, e^*_{MK1}\right) d\zeta,
\end{aligned}\right\} \tag{B 15}$$

$$\left.\begin{aligned}
-\frac{d}{dt}\int_{\zeta_1}^{\zeta_2} \bar{d}_{MN}{}^1\,ds &= [\boldsymbol{a}^1\cdot\boldsymbol{h}^*_{MN}\times\boldsymbol{v}]_{\zeta_1}^{\zeta_2} + \int_{\zeta_1}^{\zeta_2} j^*_{MN}{}^1\,ds + \int_{\zeta_1}^{\zeta_2}\left(a_{33}^{\frac{1}{2}}\, h'_{MN}{}^1 + \sum_{K=0}^{N}\bar{\psi}_K^N\, h^*_{MK3}\right) d\zeta,\\
-\frac{d}{dt}\int_{\zeta_1}^{\zeta_2} \bar{d}_{MN}{}^2\,ds &= [\boldsymbol{a}^2\cdot\boldsymbol{h}^*_{MN}\times\boldsymbol{v}]_{\zeta_1}^{\zeta_2} + \int_{\zeta_1}^{\zeta_2} j^*_{MN}{}^2\,ds + \int_{\zeta_1}^{\zeta_2}\left(a_{33}^{\frac{1}{2}}\, h'_{MN}{}^2 - \sum_{K=0}^{M}\psi_K^M\, h^*_{KN3}\right) d\zeta,\\
-\frac{d}{dt}\int_{\zeta_1}^{\zeta_2} \bar{d}_{MN}{}^3\,ds &= \int_{\zeta_1}^{\zeta_2} j^*_{MN}{}^3\,ds + \int_{\zeta_1}^{\zeta_2}\left(a_{33}^{\frac{1}{2}}\, h'_{MN}{}^3 + \sum_{K=0}^{M}\psi_K^M\, h^*_{KN2} - \sum_{K=0}^{N}\bar{\psi}_K^N\, h^*_{MK1}\right) d\zeta.
\end{aligned}\right\} \tag{B 16}$$

In the foregoing equations

$$[f(\zeta)]_{\zeta_1}^{\zeta_2} = f(\zeta_2) - f(\zeta_1).$$

Similar equations may be derived in material form. In equations (B 13)–(B 16), and in the corresponding equations in material form, we use the following definitions

$$\left.\begin{aligned}
\hat{D}_{MN}{}^1 &= a_{33}^{\frac{1}{2}}\,\bar{d}_{MN}{}^1 = A_{33}^{\frac{1}{2}}\,\bar{D}_{MN}{}^1 = \iint \hat{D}^1\chi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,dA,\\
\hat{D}_{MN}{}^2 &= a_{33}^{\frac{1}{2}}\,\bar{d}_{MN}{}^2 = A_{33}^{\frac{1}{2}}\,\bar{D}_{MN}{}^2 = \iint \hat{D}^2\psi^M(\zeta^1)\,\bar{\chi}^N(\zeta^2)\,dA,\\
\hat{D}_{MN}{}^3 &= a_{33}^{\frac{1}{2}}\,\bar{d}_{MN}{}^3 = A_{33}^{\frac{1}{2}}\,\bar{D}_{MN}{}^3 = \iint \hat{D}^3\psi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,dA,
\end{aligned}\right\} \tag{B 17}$$

$$\left.\begin{aligned}
\hat{B}_{MN}{}^1 &= a_{33}^{\frac{1}{2}}\,b_{MN}{}^1 = A_{33}^{\frac{1}{2}}\,B_{MN}{}^1 = \iint \hat{B}^1\psi^M(\zeta^1)\,\bar{\chi}^N(\zeta^2)\,dA,\\
\hat{B}_{MN}{}^2 &= a_{33}^{\frac{1}{2}}\,b_{MN}{}^2 = A_{33}^{\frac{1}{2}}\,B_{MN}{}^2 = \iint \hat{B}^2\chi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,dA,\\
\hat{B}_{MN}{}^3 &= a_{33}^{\frac{1}{2}}\,b_{MN}{}^3 = A_{33}^{\frac{1}{2}}\,B_{MN}{}^3 = \iint \hat{B}^3\chi^M(\zeta^1)\,\bar{\chi}^N(\zeta^2)\,dA,
\end{aligned}\right\} \tag{B 18}$$

$$\left.\begin{aligned}E_{MN1} &= e^*_{MN1} = \iint E_1\,\chi^M(\zeta^1)\,\overline{\psi}^N(\zeta^2)\,\mathrm{d}A,\\ E_{MN2} &= e^*_{MN2} = \iint E_2\,\psi^M(\zeta^1)\,\overline{\chi}^N(\zeta^2)\,\mathrm{d}A,\\ E_{MN3} &= e^*_{MN3} = \iint E_3\,\psi^M(\zeta^1)\,\overline{\psi}^N(\zeta^2)\,\mathrm{d}A,\end{aligned}\right\} \tag{B 19}$$

$$\left.\begin{aligned}H_{MN1} &= h^*_{MN1} = \iint H_1\,\psi^M(\zeta^1)\,\overline{\chi}^N(\zeta^2)\,\mathrm{d}A,\\ H_{MN2} &= h^*_{MN2} = \iint H_2\,\chi^M(\zeta^1)\,\overline{\psi}^N(\zeta^2)\,\mathrm{d}A,\\ H_{MN3} &= h^*_{MN3} = \iint H_3\,\chi^M(\zeta^1)\,\overline{\chi}^N(\zeta^2)\,\mathrm{d}A,\end{aligned}\right\} \tag{B 20}$$

$$\left.\begin{aligned}\hat{J}_{MN}{}^1 &= a^{\frac{1}{2}}_{33}\,j^*_{MN}{}^1 = A^{\frac{1}{2}}_{33}\,J_{MN}{}^1 = \iint \hat{J}^1\chi^M(\zeta^1)\,\overline{\psi}^N(\zeta^2)\,\mathrm{d}A,\\ \hat{J}_{MN}{}^2 &= a^{\frac{1}{2}}_{33}\,j^*_{MN}{}^2 = A^{\frac{1}{2}}_{33}\,J_{MN}{}^2 = \iint \hat{J}^2\psi^M(\zeta^1)\,\overline{\chi}^N(\zeta^2)\,\mathrm{d}A,\\ \hat{J}_{MN}{}^3 &= a^{\frac{1}{2}}_{33}\,j^*_{MN}{}^3 = A^{\frac{1}{2}}_{33}\,J_{MN}{}^3 = \iint \hat{J}^3\psi^M(\zeta^1)\,\overline{\psi}^N(\zeta^2)\,\mathrm{d}A,\\ \hat{E}_{MN} &= a^{\frac{1}{2}}_{33}\,e_{MN} = A^{\frac{1}{2}}_{33}\,E_{MN} = \iint \hat{E}\psi^M(\zeta^1)\,\overline{\psi}^N(\zeta^2)\,\mathrm{d}A,\end{aligned}\right\} \tag{B 21}$$

$$\left.\begin{aligned}a^{\frac{1}{2}}_{33}\,e'_{MN}{}^1 &= A^{\frac{1}{2}}_{33}\,E'_{MN}{}^1 = \oint \psi^M(\zeta^1)\,\overline{\chi}^N(\zeta^2)\,e^*_3\,\mathrm{d}\zeta^1,\\ a^{\frac{1}{2}}_{33}\,e'_{MN}{}^2 &= A^{\frac{1}{2}}_{33}\,E'_{MN}{}^2 = \oint \chi^M(\zeta^1)\,\overline{\psi}^N(\zeta^2)\,e^*_3\,\mathrm{d}\zeta^2,\\ a^{\frac{1}{2}}_{33}\,e'_{MN}{}^3 &= A^{\frac{1}{2}}_{33}\,E'_{MN}{}^3 = -\oint \chi^M(\zeta^1)\,\overline{\chi}^N(\zeta^2)\,(e^*_1\,\mathrm{d}\zeta^1 + e^*_2\,\mathrm{d}\zeta^2),\end{aligned}\right\} \tag{B 22}$$

$$\left.\begin{aligned}a^{\frac{1}{2}}_{33}\,h'_{MN}{}^1 &= A^{\frac{1}{2}}_{33}\,H'_{MN}{}^1 = \oint \chi^M(\zeta^1)\,\overline{\psi}^N(\zeta^2)\,h^*_3\,\mathrm{d}\zeta^1,\\ a^{\frac{1}{2}}_{33}\,h'_{MN}{}^2 &= A^{\frac{1}{2}}_{33}\,H'_{MN}{}^2 = \oint \psi^M(\zeta^1)\,\overline{\chi}^N(\zeta^2)\,h^*_3\,\mathrm{d}\zeta^2,\\ a^{\frac{1}{2}}_{33}\,h'_{MN}{}^3 &= A^{\frac{1}{2}}_{33}\,H'_{MN}{}^3 = -\oint \psi^M(\zeta^1)\,\overline{\psi}^N(\zeta^2)\,(h^*_1\,\mathrm{d}\zeta^1 + h^*_2\,\mathrm{d}\zeta^2),\end{aligned}\right\} \tag{B 23}$$

$$\left.\begin{aligned}a^{\frac{1}{2}}_{33}\,b'_{MN} &= A^{\frac{1}{2}}_{33}\,B'_{MN} = \oint \chi^M(\zeta^1)\,\overline{\chi}^N(\zeta^2)\,(\hat{B}^1\,\mathrm{d}\zeta^2 - \hat{B}^2\,\mathrm{d}\zeta^1),\\ a^{\frac{1}{2}}_{33}\,d'_{MN} &= A^{\frac{1}{2}}_{33}\,D'_{MN} = \oint \psi^M(\zeta^1)\,\overline{\psi}^N(\zeta^2)\,(\hat{D}^1\,\mathrm{d}\zeta^2 - \hat{D}^2\,\mathrm{d}\zeta^1).\end{aligned}\right\} \tag{B 24}$$

In addition

$$\left.\begin{aligned} &\boldsymbol{e}^*_{MN} = e^*_{MNi}\boldsymbol{a}^i, \quad \boldsymbol{h}^*_{MN} = h^*_{MNi}\boldsymbol{a}^i, \quad \boldsymbol{b}_{MN} = b_{MN}{}^i\boldsymbol{a}_i, \\ &\bar{\boldsymbol{d}}_{MN} = \bar{d}_{MN}{}^i\boldsymbol{a}_i, \quad \boldsymbol{j}^*_{MN} = j^*_{MN}{}^i\boldsymbol{a}_i, \\ &\boldsymbol{E}_{MN} = E_{MNi}\boldsymbol{A}^i, \quad \boldsymbol{H}_{MN} = H_{MNi}\boldsymbol{A}^i, \quad \boldsymbol{B}_{MN} = B_{MN}{}^i\boldsymbol{A}_i, \\ &\bar{\boldsymbol{D}}_{MN} = \bar{D}_{MN}{}^i\boldsymbol{A}_i, \quad \boldsymbol{J}_{MN} = J_{MN}{}^i\boldsymbol{A}_i, \\ &\boldsymbol{E}_{MN} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{e}^*_{MN}, \quad \boldsymbol{H}_{MN} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{h}^*_{MN}, \quad \Gamma\boldsymbol{b}_{MN} = \boldsymbol{F}\boldsymbol{B}_{MN}, \\ &\Gamma\bar{\boldsymbol{d}}_{MN} = \boldsymbol{F}\boldsymbol{D}_{MN}, \quad \Gamma\boldsymbol{j}^*_{MN} = \boldsymbol{F}\boldsymbol{J}_{MN}, \quad \Gamma e_{MN} = E_{MN}, \\ &\boldsymbol{F} = \boldsymbol{a}_i \otimes \boldsymbol{A}^i, \quad \Gamma = a^{\frac{1}{2}}_{33}/A^{\frac{1}{2}}_{33} \end{aligned}\right\} \tag{B 25}$$

and $\boldsymbol{a}_i$, $\boldsymbol{a}^i$ are the system of orthogonal vectors and their duals defined in (2.6) with $\boldsymbol{A}_i$, $\boldsymbol{A}^i$ the corresponding vectors in the reference configuration. Also, $\boldsymbol{v}$ is defined in (2.7).

Finally, in this appendix, we record some results which arise when the position vector, velocity vector and electromagnetic vectors have the representations (B 1) and

$$\left.\begin{aligned} \boldsymbol{e}^*\cdot\boldsymbol{g}_1 = \boldsymbol{E}\cdot\boldsymbol{G}_1 &= \sum_{K=0}^{L} \chi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,\tilde{E}_{MN1}, \\ \boldsymbol{e}^*\cdot\boldsymbol{g}_2 = \boldsymbol{E}\cdot\boldsymbol{G}_2 &= \sum_{K=0}^{L} \psi^M(\zeta^1)\,\bar{\chi}^N(\zeta^2)\,\tilde{E}_{MN2}, \\ \boldsymbol{e}^*\cdot\boldsymbol{g}_3 = \boldsymbol{E}\cdot\boldsymbol{G}_3 &= \sum_{K=0}^{L} \psi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,\tilde{E}_{MN3}, \\ \boldsymbol{h}^*\cdot\boldsymbol{g}_1 = \boldsymbol{H}\cdot\boldsymbol{G}_1 &= \sum_{K=0}^{L} \psi^M(\zeta^1)\,\bar{\chi}^N(\zeta^2)\,\tilde{H}_{MN1}, \\ \boldsymbol{h}^*\cdot\boldsymbol{g}_2 = \boldsymbol{H}\cdot\boldsymbol{G}_2 &= \sum_{K=0}^{L} \chi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,\tilde{H}_{MN2}, \\ \boldsymbol{h}^*\cdot\boldsymbol{g}_3 = \boldsymbol{H}\cdot\boldsymbol{G}_3 &= \sum_{K=0}^{L} \chi^M(\zeta^1)\,\bar{\chi}^N(\zeta^2)\,\tilde{H}_{MN3}, \end{aligned}\right\} \tag{B 26}$$

for $M+N = K$. The coefficients $\tilde{E}_{MNi}$, $\tilde{H}_{MNi}$ are related to the quantities defined in (B 19) and (B 20) by the equations

$$E_{MNi} = \sum_{R+S=0}^{L} I^{MNRS}_{(i)}\,\tilde{E}_{RSi}, \quad H_{MNi} = \sum_{R+S=0}^{L} K^{MNRS}_{(i)}\,\tilde{H}_{RSi}, \tag{B 27}$$

where

$$\left.\begin{aligned} I^{MNRS}_{(1)} &= \iint \chi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,\chi^R(\zeta^1)\,\bar{\psi}^S(\zeta^2)\,\mathrm{d}A = K^{MNRS}_{(2)}, \\ I^{MNRS}_{(2)} &= \iint \psi^M(\zeta^1)\,\bar{\chi}^N(\zeta^2)\,\psi^R(\zeta^1)\,\bar{\chi}^S(\zeta^2)\,\mathrm{d}A = K^{MNRS}_{(1)}, \\ I^{MNRS}_{(3)} &= \iint \psi^M(\zeta^1)\,\bar{\psi}^N(\zeta^2)\,\psi^R(\zeta^1)\,\bar{\psi}^S(\zeta^2)\,\mathrm{d}A, \\ K^{MNRS}_{(3)} &= \iint \chi^M(\zeta^1)\,\bar{\chi}^N(\zeta^2)\,\chi^R(\zeta^1)\,\bar{\chi}^S(\zeta^2)\,\mathrm{d}A. \end{aligned}\right\} \tag{B 28}$$

From (A 11), (B 26) and (B 4) it follows that

$$\iint \mu(w_{\mathrm{e}}^{*}+\tfrac{1}{2}\boldsymbol{\Gamma}_{\mathrm{e}}^{*}\cdot\boldsymbol{L}^{*})\,\mathrm{d}A = P_{\mathrm{e}}+\sum_{M+N=0}^{L}(\hat{J}_{MN}{}^{i}\,\tilde{E}_{MNi}+\tilde{E}_{MNi}\,\dot{\hat{D}}_{MN}{}^{i}+\tilde{H}_{MNi}\,\dot{\hat{B}}_{MN}{}^{i}), \tag{B 29}$$

where
$$P_{\mathrm{e}} = \boldsymbol{n}_{\mathrm{e}}\cdot\partial\boldsymbol{v}/\partial\zeta+\boldsymbol{k}_{\mathrm{e}}^{\alpha}\cdot\boldsymbol{w}_{\alpha}+\boldsymbol{m}_{\mathrm{e}}^{\alpha}\cdot\partial\boldsymbol{w}_{\alpha}/\partial\zeta \tag{B 30}$$

and $\boldsymbol{n}_{\mathrm{e}}, \ldots$ are related to the three-dimensional electromagnetic stress vector $\boldsymbol{T}_{\mathrm{e}}$ by formulae of the form (B 4) and (A 11). Also,

$$\iint \mu\boldsymbol{c}_{\mathrm{e}}^{*}\,\mathrm{d}A = \boldsymbol{a}_{3}\times\boldsymbol{n}_{\mathrm{e}}+\boldsymbol{d}_{\alpha}\times\boldsymbol{k}_{\mathrm{e}}^{\alpha}+\frac{\partial\boldsymbol{d}_{\alpha}}{\partial\zeta}\times\boldsymbol{m}_{\mathrm{e}}^{\alpha}. \tag{B 31}$$

Appendix C

We record in this appendix some results for the linear three-dimensional theory of a magnetic polarized thermoelastic solid, which will be of help in identifying constitutive coefficients arising in rod theory. The solid, in its reference state is homogeneous, at constant temperature $\bar{\theta}$, is unstressed and free from electromagnetic fields, but is anisotropic. We use rectangular Cartesian axes x_i along a constant orthonormal system of vectors $\boldsymbol{e}_i$, and use standard vector and Cartesian tensor notation throughout this appendix. In the linearized theory, temperature is specified by $\bar{\theta}+\theta^{*}$, ρ^{*} is reference density, $\boldsymbol{u}^{*} = u_i^{*}\boldsymbol{e}_i$ is infinitesimal displacement vector, $e_{ij}^{*} = \frac{1}{2}(u_{i,j}^{*}+u_{j,i}^{*})$ is infinitesimal strain, t_{ij} is symmetric stress tensor, η^{*} is entropy density, ψ^{*} is free energy density, ξ^{*} is internal rate of production of entropy density, $\boldsymbol{p}^{*} = p_i^{*}\boldsymbol{e}_i$ is entropy flux vector and $\boldsymbol{E} = E_i\boldsymbol{e}_i$, $\boldsymbol{H} = H_i\boldsymbol{e}_i$, $\bar{\boldsymbol{D}} = \bar{D}_i\boldsymbol{e}_i$, $\boldsymbol{B} = B_i\boldsymbol{e}_i$, $\boldsymbol{J} = J_i\boldsymbol{e}_i$, E are electromagnetic variables.

The constitutive relations in the linearized theory are given by

$$\left.\begin{aligned}
\rho^{*}\psi^{*} &= \tfrac{1}{2}c_{ijrs}\,e_{ij}^{*}\,e_{rs}^{*}-c_{ij}\,e_{ij}^{*}\,\theta^{*}-\tfrac{1}{2}c\theta^{*2}\\
&\quad-\tfrac{1}{2}f_{rs}\,E_r\,E_s-\tfrac{1}{2}g_{rs}\,H_r\,H_s-h_{rs}\,E_r\,H_s\\
&\quad-k_{rst}\,e_{rs}^{*}\,E_t-l_{rst}\,e_{rs}^{*}\,H_t+f_r\,E_r\,\theta^{*}+g_r\,H_r\,\theta^{*},\\
t_{ij} &= c_{ijrs}\,e_{rs}^{*}-c_{ij}\,\theta^{*}-k_{ijt}\,E_t-l_{ijt}\,H_t,\\
\rho^{*}\eta^{*} &= c_{ij}\,e_{ij}^{*}-f_r\,E_r-g_r\,H_r+c\theta^{*},
\end{aligned}\right\} \tag{C 1a}$$

$$\left.\begin{aligned}
\bar{D}_r &= f_{rs}\,E_s+h_{rs}\,H_s+k_{ijr}\,e_{ij}^{*}-f_r\,\theta^{*},\\
B_r &= g_{rs}\,H_s+h_{sr}\,E_s+l_{ijr}\,e_{ij}^{*}-g_r\,\theta^{*},\\
p_i^{*} &= -k_{ij}\,\theta_{,j}^{*}-\bar{a}_{ij}\,E_j,\\
J_i &= l_{ij}\,\theta_{,j}^{*}+b_{ij}\,E_j,
\end{aligned}\right\} \tag{C 1b}$$

$$J_i\,E_i-p_i^{*}\,\theta_{,i}^{*} = \rho^{*}(\bar{\theta}+\theta^{*})\,\xi^{*} \geqslant 0, \tag{C 2}$$

In (C 1) and (C 2), the notation $(\ )_{,i}$ denotes partial differentiation with respect to x_i, and the various coefficients are constants and subject to the following restrictions:

$$\left.\begin{aligned}
&c_{ijrs} = c_{jirs} = c_{ijsr} = c_{rsij}, \quad c_{ij} = c_{ji},\\
&f_{rs} = f_{sr}, \quad g_{rs} = g_{sr}, \quad k_{rst} = k_{srt}, \quad l_{rst} = l_{srt}.
\end{aligned}\right\} \tag{C 3}$$

In making use of the above results, it is convenient to express them in a partially inverted form, i.e.

$$\left.\begin{aligned} e^*_{ij} &= s_{ijrs}\, t_{rs} + s_{ij}\,\theta^* - k^*_{ijt}\, E_t - l^*_{ijt}\, H_t, \\ \rho^*\eta^* &= s_{ij}\, t_{ij} - f^*_r\, E_r - g^*_r\, H_r + c^*\theta^*, \\ \bar{D}_r &= -k^*_{ijr}\, t_{ij} - f^*_r\,\theta^* - f^*_{rs}\, E_s - h^*_{rs}\, H_s, \\ B_r &= -l^*_{ijr}\, t_{ij} - g^*_r\,\theta^* - h^*_{sr}\, E_s - g^*_{rs}\, H_s, \end{aligned}\right\} \tag{C 4}$$

where the coefficients have symmetry restrictions similar to the corresponding coefficients in (C 3). Also,

$$\left.\begin{aligned} & c_{ijrs}\, s_{rsmn} = \tfrac{1}{2}(\delta_{im}\,\delta_{jn} + \delta_{in}\,\delta_{jm}), \\ & c_{ijrs}\, s_{rs} - c_{ij} = 0, \quad c - c^* + c_{ij}\, s_{ij} = 0, \\ & c_{ijrs}\, k^*_{rst} + k_{ijt} = 0, \quad c_{ijrs}\, l^*_{rst} + l_{ijt} = 0, \\ & f^*_r = f_r - s_{ij}\, k_{ijr}, \quad g^*_r = g_r - s_{ij}\, l_{ijr}, \\ & f^*_{rs} + f_{rs} = k_{ijr}\, k^*_{ijs}, \quad h^*_{rs} + h_{rs} = k_{ijr}\, l^*_{ijs}, \\ & g^*_{rs} + g_{rs} = l_{ijr}\, l^*_{ijs}, \end{aligned}\right\} \tag{C 5}$$

with similar formulae in which c_{ijrs}, c_{ij} are interchanged with s_{ijrs}, s_{ij}, respectively, and in which starred and unstarred quantities are interchanged.

References

Green, A. E. & Laws, N. 1966 A general theory of rods. *Proc. R. Soc. Lond.* A **283**, 145.

Green, A. E. & Laws, N. 1973 Remarks on the theory of rods. *J. Elasticity* **3**, 179.

Green, A. E. & Naghdi, P. M. 1977 On thermodynamics and the nature of the second law. *Proc. R. Soc. Lond.* A **357**, 253.

Green, A. E. & Naghdi, P. M. 1979 On thermal effects in the theory of rods. *Int. J. Solids Struct.* **15**, 829.

Green, A. E. & Naghdi, P. M. 1983 On electromagnetic effects in the theory of shells and plates. *Phil. Trans. R. Soc. Lond.* A **309**, 559.

Green, A. E. & Naghdi, P. M. 1984 Electromagnetic effects in an elastic circular cylindrical waveguide. *J. math. Phys.* **25**, 2461.

Green, A. E., Naghdi, P. M. & Wenner, M. L. 1974 On the theory of rods II. Development by direct approach. *Proc. R. Soc. Lond.* A **337**, 485.

Hutter, K. & van de Ven, A. A. F. 1978 Field matter interactions in thermoelastic solids. *Lect. Notes Phys.* no. 88. Berlin: Springer-Verlag.

Mindlin, R. D. 1976 *Report of workshop on 'Applications of elastic waves in electrical devices, non-destructive testing and seismology'* (ed. J. D. Achenback, Y. H. Pao & H. F. Tiersten), p. 24. Natural Science Foundation.

Naghdi, P. M. 1979 On the applicability of directed fluid jets to Newtonian and non-Newtonian flows. *J. Non-Newtonian Fluid Mech.* **5**, 233.

Naghdi, P. M. 1982 Finite deformation of elastic rods and shells. In *Proc. IUTAM Symp. on Finite Elasticity* (ed. D. E. Carlson & R. T. Shield), p. 47. The Hague, The Netherlands: Martinus Nijhoff.

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